Lefschetz properties and Rees algebras of squarefree monomial ideals

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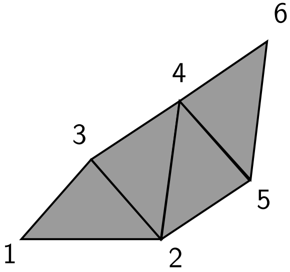
Stanley-Reisner, Facet (and incidence) ideals

A simplicial complex Δ on vertex set $[n]$ is a collection of subsets Δ of $[n]$ such that $\tau \subset \sigma \in \Delta \implies \tau \in \Delta$. We write $\Delta = \langle F_1, \ldots, F_s \rangle$ if $F_1, \ldots, F_s$ are the facets (maximal subsets) of Δ.



The simplicial complex $\Delta = \langle \{1,2,3\}, \{2,3,4\}, \{2,4,5\}, \{5,4,6\} \rangle$

If we remove every 2-face of Δ (i.e the triangles), we get the complex $\Delta(1)$ which consists of the same vertices and edges of Δ, but no triangles

Stanley-Reisner, Facet (and incidence) ideals

Let $S = k[x_1, \ldots, x_n]$ and $\Delta = \langle F_1, \ldots, F_s \rangle$ a simplicial complex with vertex set $[n]$.

- The **Stanley-Reisner** ideal of Δ is the ideal

$$\mathcal{N}(\Delta) = (\prod_{i \in B} x_i : B \notin \Delta) \subset S$$

- The **Facet** ideal of Δ is the ideal

$$\mathcal{F}(\Delta) = (\prod_{i \in F_1} x_i, \ldots, \prod_{i \in F_s} x_i) \subset S$$

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Both constructions give bijections between simplicial complexes and squarefree monomial ideals

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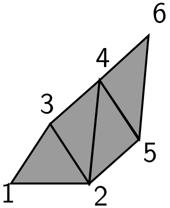
$(x_1x_4, x_1x_5, x_3x_5, x_1x_6, x_2x_6, x_3x_6)$ ↔ Δ

$\mathcal{N}(\Delta)$



Δ

↕

$(x_1x_2x_3, x_2x_3x_4, x_2x_4x_5, x_4x_5x_6)$

$\mathcal{F}(\Delta)$

Lefschetz properties

Let I be a homogeneous ideal of $S = k[x_1, \ldots, x_n]$ such that $A = S/I$ is artinian, and $L \in S_1$ a general linear form.

Definition

We say A satisfies the **weak Lefschetz property (WLP)** if the multiplication maps

$$\times L : A_i \to A_{i+1}$$

have full rank for every i.
If moreover the maps

$$\times L^j : A_i \to A_{i+j}$$

have full rank for every i, j, we say A satisfies the **strong Lefschetz property (SLP)**

A motivation from Combinatorics

Proposition

If A is an algebra that satisfies the WLP, then

$$\dim A_1 \leq \dim A_2 \leq \cdots \leq \dim A_k \geq \cdots \geq \dim A_d$$

for some k, in other words, the h-vector of A is unimodal.

A motivation from Combinatorics

Proposition

If A is an algebra that satisfies the WLP, then

$$\dim A_1 \leq \dim A_2 \leq \cdots \leq \dim A_k \geq \cdots \geq \dim A_d$$

for some k, in other words, the h-vector of A is unimodal.

We are particularly interested in algebras of the form:

$$A(\Delta) = \frac{S}{(\mathcal{N}(\Delta), x_1^2, \ldots, x_n^2)}$$

where Δ is a simplicial complex.

Some known properties of $A(\Delta)$

Useful facts

Let Δ be a simplicial complex.

- Non zero monomials in $A(\Delta)$ are in bijection with faces of Δ

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Useful facts

Let Δ be a simplicial complex.

- Non zero monomials in $A(\Delta)$ are in bijection with faces of Δ
- $\dim A(\Delta)_i = f_{i-1} =$ number of $i-1$ dimensional faces of Δ
- Since $A(\Delta)$ is the quotient by a monomial ideal, $A(\Delta)$ has the WLP/SLP if and only if the maps

$$\times L^j : A(\Delta)_i \to A(\Delta)_{i+j}, \quad L = x_1 + \cdots + x_n$$

  have full rank

The WLP of $A(\Delta)$ is a sufficient condition for the unimodality of the f-vector of Δ $(f_{-1}, f_0, \ldots, f_d)$

An example with the SLP

Let $\mathcal{N}(\Delta) = (x_1x_4, x_1x_5, x_3x_5, x_1x_6, x_2x_6, x_3x_6) \subset S = k[x_1, \ldots, x_6]$, Then

$$A(\Delta) = \frac{S}{(\mathcal{N}(\Delta), x_1^2, \ldots, x_6^2)}$$

and

$$A(\Delta)_1 \xrightarrow{\begin{array}{c|cccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \hline x_1x_2 & 1 & 1 & 0 & 0 & 0 & 0 \\ x_1x_3 & 1 & 0 & 1 & 0 & 0 & 0 \\ x_2x_3 & 0 & 1 & 1 & 0 & 0 & 0 \\ x_2x_4 & 0 & 1 & 0 & 1 & 0 & 0 \\ x_2x_5 & 0 & 1 & 0 & 0 & 1 & 0 \\ x_3x_4 & 0 & 0 & 1 & 1 & 0 & 0 \\ x_4x_5 & 0 & 0 & 0 & 1 & 1 & 0 \\ x_4x_6 & 0 & 0 & 0 & 1 & 0 & 1 \\ x_5x_6 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}} A(\Delta)_2$$

has full rank in every odd characteristic

An example with the SLP

Let $\mathcal{N}(\Delta) = (x_1x_4, x_1x_5, x_3x_5, x_1x_6, x_2x_6, x_3x_6) \subset S = k[x_1, \dots, x_6]$, Then

$$A = A(\Delta) = \frac{S}{(\mathcal{N}(\Delta), x_1^2, \dots, x_6^2)}$$

and

$$A_2 \xrightarrow{\begin{array}{c} \quad\; \begin{array}{ccccccccc} x_1x_2 & x_1x_3 & x_2x_3 & x_2x_4 & x_2x_5 & x_3x_4 & x_4x_5 & x_4x_6 & x_5x_6 \end{array} \\ \begin{array}{c} x_1x_2x_3 \\ x_2x_3x_4 \\ x_2x_4x_5 \\ x_4x_5x_6 \end{array} \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}} A_3$$

has full rank in every characteristic.
$\times L^2 : A_1 \to A_3$ also has full rank in every characteristic, so A has the SLP in every odd characteristic.

Incidence matrices (everywhere!)

The two matrices that represent the maps we just saw have very particular structures:

$$
A(\Delta)_1 \xrightarrow{\begin{array}{c c} & \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{array}{c} x_1x_2 \\ x_1x_3 \\ x_2x_3 \\ x_2x_4 \\ x_2x_5 \\ x_3x_4 \\ x_4x_5 \\ x_4x_6 \\ x_5x_6 \end{array} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}} A(\Delta)_2
$$

Taking rows as exponents we have the ideal

$$\mathcal{F}(\Delta(1)) = (x_1x_2, x_1x_3, x_2x_4, x_2x_4, x_2x_5, x_3x_4, x_4x_5, x_4x_6, x_5x_6)$$

where $\Delta(1)$ is the simplicial complex where the facets are the 1-faces of Δ

Incidence matrices (everywhere!)

We call the matrices that represent the multiplication by L maps in $A(\Delta)$ the **incidence matrices** of Δ.
Taking rows as exponents we have the **incidence ideals** of Δ. Incidence ideals are ideals in the **incidence ring** of Δ:

$$S_\Delta = \mathbb{C}[x_\tau : \tau \in \Delta]$$

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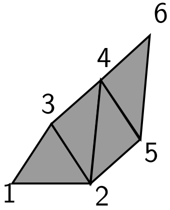
Useful fact

The incidence ideals of Δ corresponding to maps $\times L^{i-1} : A_1 \to A_i$ are the facet ideals of skeletons, i.e $\mathcal{F}(\Delta(i))$

Incidence matrices (everywhere)

- $\times L : A(\Delta)_1 \to A(\Delta)_2$ corresponds to the ideal $(x_1x_2, x_1x_3, x_2x_4, x_2x_4, x_2x_5, x_3x_4, x_4x_5, x_4x_6, x_5x_6)$
- $\times L : A(\Delta)_2 \to A(\Delta)_3$ corresponds to the ideal $(x_{\{1,2\}}x_{\{1,3\}}x_{\{2,3\}}, x_{\{2,3\}}x_{\{2,4\}}x_{\{3,4\}}, x_{\{2,4\}}x_{\{2,5\}}x_{\{4,5\}}, x_{\{4,5\}}x_{\{4,6\}}x_{\{5,6\}})$



Δ

The bipartite property in Combinatorial Commutative Algebra

Graph theory result

The incidence matrix of a connected graph with more edges than vertices has full rank if and only if it is not bipartite.

The bipartite property in Combinatorial Commutative Algebra

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Birational monomial maps

A rational monomial map $\mathbb{P}^n \to \mathbb{P}^n$ defined by the edge ideal of a graph is birational if and only if the graph is not bipartite (SV, 2005)

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WLP

- The map $\times L : A(\Delta)_1 \to A(\Delta)_2$ where Δ is connected has full rank in char 0 if and only if $\Delta(1)$ is not bipartite (DN, 2021)

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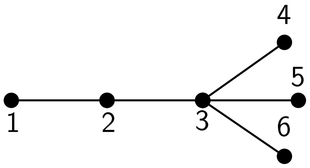
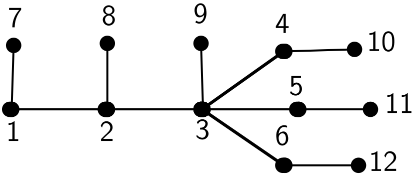
- The map $\times L : A(\Delta)_1 \to A(\Delta)_2$ where Δ is connected has full rank in char 0 if and only if $\Delta(1)$ is not bipartite (DN, 2021)
- The map $\times L : A_1 \to A_2$ where A is any artinian monomial algebra has full rank in positive odd characteristic if and only if it does so in char 0 (-, 2023)

A family of examples: whiskered graphs



B_3



$w(B_3)$

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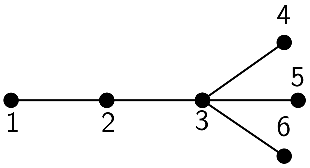

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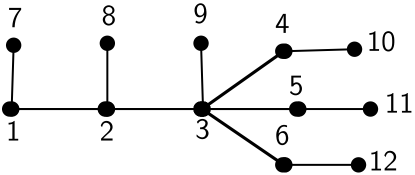

$w(B_3)$

Theorem [Cooper,Faridi,-,Nicklasson,Van Tuyl 2023]

If $I = \mathcal{N}(\Delta)$ is the edge ideal of a whiskered graph on $2n$ vertices with at least $n+1$ edges over a field of characteristic zero, then the maps $\times L : A(\Delta)_i \to A(\Delta)_{i+1}$ have full rank for $i < n/2$ and $i = n-1$. Moreover, the first and last maps have full rank if the characteristic is not 2.

This result is optimal: the graph above is an example where these maps are the only ones that have full rank.

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Graph theory result

Incidence matrix has full rank $\iff$ graph is not bipartite

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Symbolic powers

The symbolic powers of an edge ideal of a graph are equal to the ordinary powers if and only if the graph is **not** bipartite. (SVV, 1994)

The bipartite property in Combinatorial Commutative Algebra

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Incidence matrix has full rank $\iff$ graph is not bipartite

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The symbolic powers of an edge ideal of a graph are equal to the ordinary powers if and only if the graph is **not** bipartite. (SVV, 1994)

Linear type

The edge ideal of a connected graph with the same number of vertices and edges is of linear type if and only if it is not bipartite. (V, 1995)

An ideal is said to be of linear type if its Rees algebra is naturally isomorphic to its Symmetric algebra

The bipartite property in Combinatorial Commutative Algebra

$$\begin{aligned} \text{Not bipartite} &\iff \text{Birational maps} \\ &\iff \text{Linear type} \\ &\iff \text{Symbolic powers do } \textbf{not} \text{ match} \\ &\iff \text{Incidence matrix has full rank (WLP)} \end{aligned}$$

The bipartite property in Combinatorial Commutative Algebra

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&\iff \text{Incidence matrix has full rank (WLP)}
\end{aligned}
$$

But what can we say for simplicial complexes in general?

$\mathcal{F}(\Delta)$ Rees $\implies \mathcal{N}(\Delta)$ Lefschetz

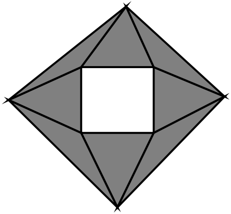
Corollary

If Δ is connected and pure of dimension 2, then:

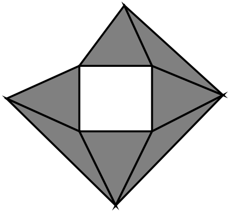
$$\mathcal{F}(\Delta) \text{ is of linear type } \implies A(\Delta) \text{ has the SLP}$$

The result above is an example of information on the Rees algebra of $\mathcal{F}(\Delta)$ being translated into information on the Lefschetz properties of $\mathcal{N}(\Delta)$ (or more specifically, $A(\Delta)$)

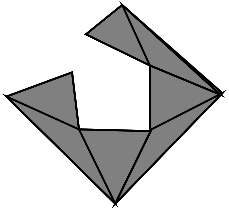
From linear type to Lefschetz properties: sufficient conditions visualized



Linear type results can't be used



Linear type results imply WLP in every odd characteristic



SLP in every odd characteristc

Symbolic powers

Symbolic powers of squarefree monomial ideals

Let $\mathcal{F}(\Delta) \subset S = k[x_1, \ldots, x_n]$ be a squarefree monomial ideal. The m-th symbolic power of $\mathcal{F}(\Delta)$ is:

$$\mathcal{F}(\Delta)^{(m)} = \bigcap_{P \in \mathrm{Ass(I)}} P^m$$

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If $\mathcal{F}(\Delta) = (x_1x_2, x_2x_3, x_1x_3)$, then

$$\mathcal{F}(\Delta)^{(2)} = (x_1x_2x_3, x_1^2x_2^2, x_2^2x_3^2, x_1^2x_3^2) \neq \mathcal{F}(\Delta)^2$$

Symbolic Powers and Lefschetz properties

SLP and Symbolic powers are not compatible

If Δ is a pure simplicial complex with at least as many facets as vertices such that $\mathcal{F}(\Delta)^{(m)} = \mathcal{F}(\Delta)^m$ for all m, then $A(\Delta)$ fails the SLP in characteristic 0.

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If Δ is a pure simplicial complex with at least as many facets as vertices such that $\mathcal{F}(\Delta)^{(m)} = \mathcal{F}(\Delta)^m$ for all m, then $A(\Delta)$ fails the SLP in characteristic 0.

Note that the result above is another example of a result of the form:

$$\mathcal{F}(\Delta) \text{ Symbolic powers (Rees)} \implies \mathcal{N}(\Delta) \text{ Lefschetz}$$

where information on the Rees algebra of $\mathcal{F}(\Delta)$ translates into information on the Lefschetz properties of $\mathcal{N}(\Delta)$, or more specifically, $A(\Delta)$.

The symbolic defect (polynomials)

Symbolic Defect sequence of an ideal (GGSVT, 2018)

Let I be an ideal, define $\operatorname{sdefect}(I, m) = \mu(I^{(m)}/I^m)$ for every m.

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Let I be an ideal, define $\mathrm{sdefect}(I, m) = \mu(I^{(m)}/I^m)$ for every m.

GGSVT, 2018

If I is the ideal generated by every squarefree monomial ideal of degree d in n variables, then

$$\mathrm{sdefect}(I, 2) = \binom{n}{d+1}$$

In other words, $\mathrm{sdefect}(\mathcal{F}(\Delta(d)), 2) = \binom{n}{d+1}$, where Δ is the simplex in n variables.

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Second symbolic defect of an edge ideal

Let G be a graph with t triangles and edge ideal I. Then

$$\operatorname{sdefect}(I, 2) = t$$

Symbolic defect polynomials

Instead of looking at sequences sdefect(I, m) with m varying, we can look at sequences sdefect($\mathcal{F}(\Delta(i)), 2$) with i varying.

The second symbolic defect polynomial

The **second symbolic defect polynomial** of a pure simplicial complex Δ is:

$$\mu(\Delta, 2, x) = \sum_i \text{sdefect}(\mathcal{F}(\Delta(i)), 2) x^{i+2}$$

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The second symbolic defect polynomial

The **second symbolic defect polynomial** of a pure simplicial complex Δ is:

$$\mu(\Delta, 2, x) = \sum_{i} \text{sdefect}(\mathcal{F}(\Delta(i)), 2) x^{i+2}$$

- If Δ is a simplex on n vertices, then $\mu(\Delta, 2, x) = (1+x)^n - 1 - nx - \binom{n}{2} x^2$
- The coefficient of x^3 in $\mu(\Delta, 2, x)$ is always equal to the number of triangles of Δ.
- The sequence of coefficients of $\mu(\Delta, 2, x)$ has no internal zeros.

A couple of examples

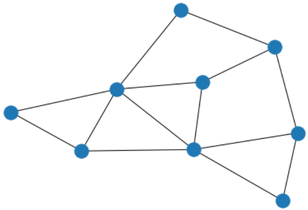
Let $\mathcal{N}(\Delta) = (x_i x_{i+1} : 1 \leq i \leq 14) \subset k[x_1, \ldots, x_{15}]$. Then

$$\mu(\Delta, 2, x) = 286x^3 + 495x^4 + 462x^5 + 210x^6 + 36x^7 + x^8$$

and the f-vector of Δ is:

$$(1, 15, 91, 286, 495, 462, 210, 36, 1)$$

A couple of examples



The Stanley-Reisner complex Δ of the edge ideal of the graph above has

- $\mu(\Delta, 2, x) = 17x^3 + 5x^4$
- f-vector: $(1, 9, 22, 17, 4)$

So the two are not always the same

Unimodality? Log-concavity? f-vectors?

Questions

- When is the second symbolic defect polynomial of a complex unimodal?
- When is the second symbolic defect polynomial of a complex equal to its f-vector?