Homological invariants of ternary graphs

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Independence Complexes

Given a graph $G = (V, E)$, we define its edge ideal

$$I(G) := (x_i x_j | \{i, j\} \in E)$$

and given a simplicial complex Δ, we define its Stanley-Reisner ideal

$$I_\Delta := (x_{i_1} \dots x_{i_s} | \{i_1, \dots, i_s\} \notin \Delta)$$

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Theorem (Hochster's formula)

Let Δ be a simplicial complex. Then

$$b_{i, x_\tau}(I_\Delta) = \dim \tilde{H}_{|\tau|-i-2}(\Delta_\tau; k)$$

where Δ_τ is the restriction of Δ

Independence complexes

Let $S = k[x_1, \ldots, x_n]$, $I(G)$ the edge ideal of a graph G and I_Δ the Stanley-Reisner ideal of Δ.

Independence complex of G

A set $S \subset V(G)$ is a face of the simplicial complex $\mathrm{Ind}(G)$ if and only if S is an independent set of G, that is, none of the edges of G are between elements of S.

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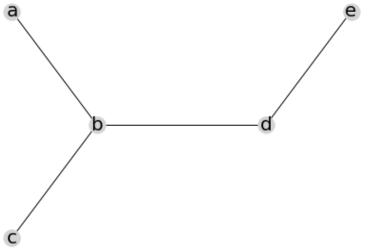
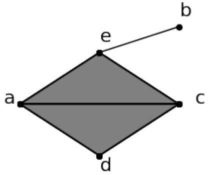
Useful facts

- If G has an isolated vertex, $\mathrm{Ind}(G)$ is a cone.
- $I(G) = I_{\mathrm{Ind}(G)}$

Independence complexes



(a) A graph G



(b) Ind(G)

Edge covering complexes

Let G be a graph without isolated vertices. An edge covering of G is a set of edges T such that every vertex is an endpoint of an edge of T.

Edge covering complex

The vertices of EC_G are the edges of G. The minimal nonfaces of the simplicial complex EC_G are the minimal edge coverings of G.

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Another useful fact

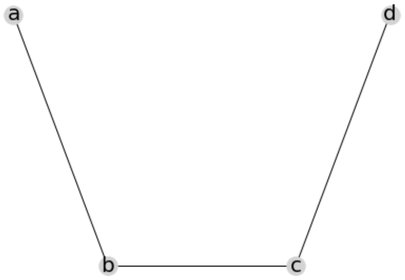
Let $\Theta_{<m}$ be the restriction of the Taylor complex of $I(G)$ to $m = \prod_{v \in V} x_v$.

Then

$$\Theta_{<m} \cong EC_G.$$

In particular, EC_G is contractible if and only if $\text{Ind}(G)$ is contractible

Edge covering complexes



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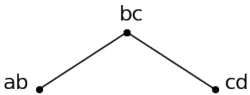

(b) EC_G (or $\Theta_{<abc}$)

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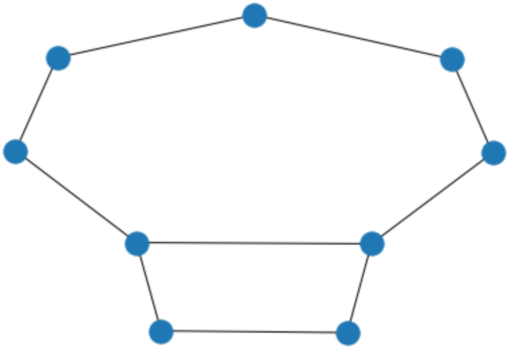

A ternary graph with a non-induced 9-cycle

Ternary graphs

Theorem (J. Kim, 2022)

A graph is ternary if and only if $\mathrm{Ind}(G)$ *is either contractible or homotopy equivalent to a sphere for every induced subgraph* G.

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Corollary

The betti table of the edge ideal of a ternary graph does not depend on the characteristic of the base field.

Ternary graphs

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Ternary graphs

Let G be a ternary graph.

1. When is $\mathrm{Ind}(G)$ contractible?
2. When $\mathrm{Ind}(G)$ is not contractible, what is the dimension of the sphere $\mathrm{Ind}(G)$ is homotopy equivalent to?
3. Can we describe projective dimension, depth and regularity of $S/I(G)$ in terms of G? (these invariants are characteristic-free)

Setting some notation

Given a graph G and an independent subset $X \subset V(G)$, we set
$N[X] = \bigcup_{v \in X} N(v) \bigcup_{v \in X} v$.

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One way to think about $\mathrm{Ind}(G(X|Y))$ is:
"the faces of $\mathrm{Ind}(G)$ that contain X and are disjoint from Y.

Forests

Theorem (M. Marietti and D. Testa, 2008)

Let G be a forest. Then $\mathrm{Ind}(G)$ *is either contractible or homotopy equivalent to* $S^{\gamma(G)-1}$*, where* $\gamma(G) = \min\{|S| \mid S \subset V(G),\ N[S] = V(G)\}$ *is called the lower dominating number of G.*

Lemma

Let F be a forest and v a vertex adjacent to a leaf u. Then

$$\mathrm{Ind}(F) \cong \mathrm{Ind}(F(v|\emptyset))$$

Leaf-filtrations

A *leaf-filtration* $\mathcal{F}$ of F is a sequence of induced subgraphs of F:

$$\mathcal{F} : F = F_0 \supset F_1 \supset \cdots \supset F_{q-1} \supset F_q$$

where for $i = 1, \ldots, q-1$ we have $F_i = F_{i-1}(v_{i-1}|\emptyset)$, where v_{i-1} is adjacent to a leaf of F_{i-1}.

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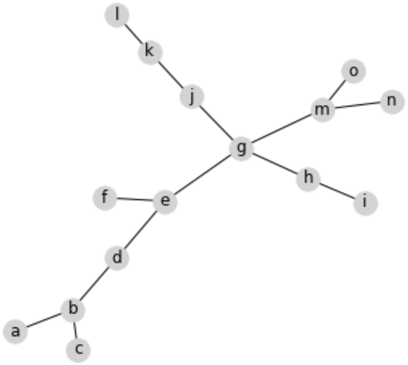
Note that $\text{Ind}(F)$ is contractible if and only if $\text{Ind}(F_i)$ is contractible for $i = 0, \ldots, q$.

A leaf-filtration

The following sequence of graphs is a leaf-filtration of the graph G below



$$G = G(\emptyset|\emptyset), G(b|\emptyset), G(b, e|\emptyset), G(b, e, k|\emptyset), G(b, e, k, m|\emptyset), G(b, e, k, m, h|\emptyset) = \emptyset$$

Edges stuck between leaves

An edge e of a graph G is said to be *stuck between leaves* if both of its endpoints are adjacent to leaves. We say e is an SBL of G.

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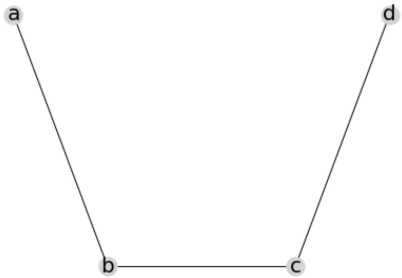
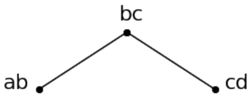
Theorem

If G has an SBL, then EC_G is a cone over the SBL. In particular, $\mathrm{Ind}(G)$ *is contractible.*

Example of SBL



(a) A graph G with an SBL bc



(b) EC_G is a cone over bc

Hidden SBLs

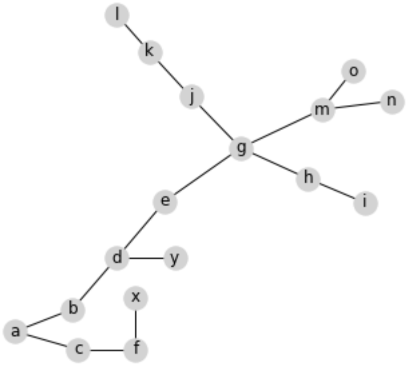
A *hidden SBL* of a forest F is an SBL of an induced subgraph F_i in a leaf-filtration of F.

Theorem

Let F be a forest without isolated vertices. The independence complex of F is contractible if and only if F has a hidden SBL.

An example of hidden SBL

The edge *bd* is a hidden SBL of the graph G below



The graph $G - N[f]$ has an SBL *bd* which is not an SBL of G

Back to ternary graphs

Let G be a ternary graph with n cycles. Let $S \subset V(G)$ be such that $G(\emptyset|S)$ is a forest. Then whenever $A \subset S$ is an independent set $G(A|S\backslash A)$ is also a forest.

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Definition (The sign of a ternary graph)

Let $j_S(G)$ be the number of forests $G(A|S \backslash A)$ that dont have hidden SBLs or isolated vertices where $A \subset S$ is an independent set. The *sign of* G is $i_S(G) = (-1)^{j_S(G)}$.

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Lemma

Given any two suitable sets S, S' we have $i_S(G) = i_{S'}(G)$.

Because of the lemma above we write $i(G)$ for the sign of G

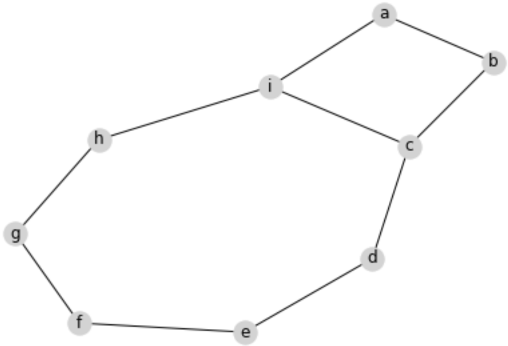
When is the independence complex of a ternary graph contractible

Theorem

The independence complex of a ternary graph G is contractible if and only if $i(G) = 1$

Example sign

Let G be the following graph and $S = \{e, b\}$



The graph $G - S$ is a forest

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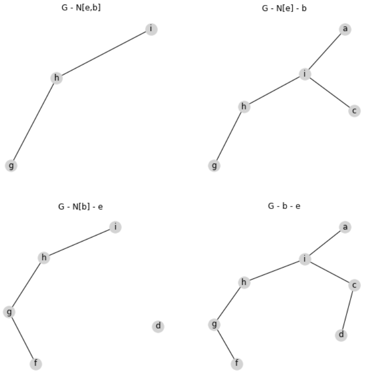

The only graph without SBLs above is $G - N[S]$, so the sign of G is -1.

Filtrations

Definition (Filtration)

A filtration $\mathcal{F}$ of a ternary graph G with negative index is a sequence of induced subgraphs:

$$\mathcal{F} : G \supset G_1 \supset \cdots \supset G_{q-1} \supset G_q = \emptyset$$

such that the homology groups of $\text{Ind}(G_i)$ are isomorphic to the homology groups of $\text{Ind}(G_{i+1})$ (possibly with a shift) and G_{i+1} is either $G_i - v_i$ or $G_i - N[v_i]$ for some $v_i \in V(G_i)$

Some notation

Let G be a ternary graph with non contractible independence complex and $\mathcal{F}$ a filtration of G.

Notation

1. The *vertex deletion number* of $\mathcal{F}$ is $\mathrm{del}(\mathcal{F}) = |\{i \mid G_i = G_{i-1} - v_i\}|$
2. The *deleted neighborhood* of $\mathcal{F}$ is $N(\mathcal{F}) = \{v_i \mid G_i = G_{i-1} - N[v_i]\}$
3. The *depth* of $\mathcal{F}$ is $\mathrm{depth}(\mathcal{F}) = |N(\mathcal{F})|$

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Theorem

The independence complex of G is homotopy equivalent to $S^{\mathrm{depth}(\mathcal{F})-1}$.

Main definitions

Let G be a ternary graph with negative index and $\mathcal{F}$ an arbitrary filtration.

1. The *projective dimension* of G is $\mathrm{pd}(G) = \mathrm{del}(\mathcal{F}) + \sum_{v \in N(\mathcal{F})} \deg v$
2. The *depth* of G is $\mathrm{depth}(G) = \mathrm{depth}(\mathcal{F}) = |V| - \mathrm{pd}(G)$

Theorem

- $\mathrm{pd}(G) = \mathrm{pd}(R/I(G))$
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