Roller-Coasters, graphs and Perazzo forms

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Gorenstein sequences and the question of characterizing them

Let $h_0, h_1, \ldots, h_{d-1}, h_d$ be the Hilbert function of some standard graded artinian algebra. What are necessary and sufficient conditions to guarantee that there exists a standard graded Artinian Gorenstein algebra A with

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- Stanley showed the sequence does not have to be unimodal $(1, 13, 12, 13, 1)$
- Boij showed using compressed algebras that they can have arbitrarily many valleys

Similar questions in Combinatorics

- Classify h and f-vectors of (flag) triangulations of spheres
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Other places where independence polynomials show up include

- Hilbert functions of quadratic artinian monomial algebras
- f-vectors of flag complexes

Independence polynomials and well-covered graphs

Let G be a graph and consider the edge ideal of G

$$I(G) = (x_i x_j \colon ij \in E(G)) \subseteq R = k[x_1, \ldots, x_n]$$

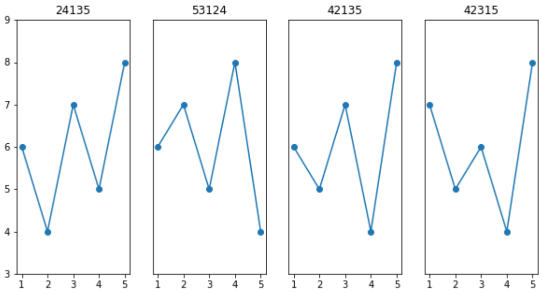
- A graph is said to be well-covered if its edge ideal is unmixed
- The independence polynomial of a graph G is the Hilbert series of the quotient by $J = I(G) + (x_1^2, \ldots, x_n^2)$. The sequence of coefficients of this polynomial is called the independent set sequence of G
- The independence number of a graph G is the socle degree of R/J

"Two unfortunate properties of (pure) f-vectors"

Theorem (Independence polynomials are unconstrained: Alavi-Malde-Schwenk-Erdos 1987)

The independent set sequence of a graph is unconstrained. In other words, for every permutation π on $[m] = \{1, \ldots, m\}$, there exists a graph G with independence number m and independent set sequence satisfying

$$h_{\pi(1)} < h_{\pi(2)} < \cdots < h_{\pi(m)}$$



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Theorem (The Roller-Coaster theorem, Cutler and Pebody (2014))

The second half of the independent set sequence of unmixed graphs is unconstrained. In other words, for every permutation π *on* $\{\lceil \frac{m}{2} \rceil, \ldots, m\}$*, there exists a graph* G *with independence number* m *and independent set sequence satisfying*

$$h_{\pi(\lceil \frac{m}{2} \rceil)} < h_{\pi(\lceil \frac{m}{2} \rceil + 1)} < \cdots < h_{\pi(m)}$$

Macaulay duality and Perazzo forms

Let $R = k[x_1, \dots, x_n]$ and $S = k[X_1, \dots, X_n]$ be two polynomial rings. Moreover, define $x_i \circ F = \frac{\partial F}{\partial X_i}$ for every $F \in S$.

Macaulay duality

In the setting above, there is a bijection between standard graded artinian Gorenstein R-algebras and homogeneous forms $F \in S$ given by:

$$R/Ann_R(F) = A_F \leftrightarrow F$$

where $Ann_R(F) = \{g \in R \colon g \circ F = 0\} \subset R$

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Perazzo forms: vanishing Hessians but not cones

A homogeneous form $F \in k[X_1, \dots, X_n, U_1, \dots, U_s]$ is called a Perazzo form if it can be written as

$$F = X_1 G_1(U_1, \dots, U_s) + \cdots + X_n G_n(U_1, \dots, U_s) + G(U_1, \dots, U_s)$$

where the G_i are linearly independent but algebraically dependent

Worse than arbitrarily many valleys

Theorem (The Roller-Coaster theorem for artinian Gorenstein algebras, CFHNVT (2024+))

For every permutation π *on* $\{1, \ldots, \lfloor \frac{m}{2} \rfloor\}$*, there exists an artinian Gorenstein algebra with Hilbert function* $1, h_1, \ldots, h_m$ *such that*

$$h_{\pi(1)} < h_{\pi(2)} < \cdots < h_{\pi(\lfloor \frac{m}{2} \rfloor)}$$

Moreover, the Macaulay dual of such algebras can be taken to be Perazzo forms

Where are they and what do they look like?

- Proof uses the same idea from the Roller-Coaster conjecture in graph theory. It is an ε, δ proof. The main concept are the so called "Approximate well-covered polynomials" introduced by Cutler and Pebody

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- Proof uses the same idea from the Roller-Coaster conjecture in graph theory. It is an ε, δ proof. The main concept are the so called "Approximate well-covered polynomials" introduced by Cutler and Pebody
- Codimension of the algebras in the proof will most likely be huge (possibly higher than 3^m)
- They come from the Nagata idealization of $R/(I(G) + (x_i^2))$, where G comes from the Roller-Coaster theorem (similar to Boij's proof of arbitrarily many valleys)

Can Koszul AG algebras be as bad? Quadratic Gröbner basis?

- D'Ali and Venturello showed that the algebras we have are Koszul if and only if the starting graph is CM, and they have a quadratic Gröbner basis if and only if the graph is "shellable".

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- Using similar methods we can also construct large families of Perazzo forms that fail the Weak Lefschetz property in many degrees
- Given a permutation π a natural question that arises is what is the minimal number n such that there exists an AG algebra with codimension n satisfying the permutation condition.