From points to complexes: a concept of unexpectedness for simplicial complexes

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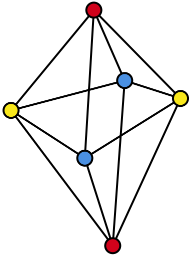
students/postdocs who speak portuguese: check out irere.org

Simplicial complexes and Stanley-Reisner ideals

A simplicial complex Δ is a collection of subsets of $[n]$ closed under inclusion. The Stanley-Reisner ideal of Δ is

$$I_\Delta = (\prod_{i\in\sigma} x_i \colon \sigma \notin \Delta) \subset R = \mathbb{K}[x_1, \dots, x_n]$$



$$I_\Delta = (x_1x_2, x_3x_4, x_5x_6)$$

The early days of combinatorial commutative algebra

First applications of commutative algebra to combinatorics: trying to classify the possible number of i-faces of polytopes/spheres (UBC, g-conjecture, etc)

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A standard graded artinian algebra A has the weak Lefschetz property (WLP) if there exists a linear form $\ell \in A_1$ such that $\times \ell : A_i \to A_{i+1}$ always has full rank

The idea: the WLP for $R/(I_\Delta, \theta_1, \ldots, \theta_d)$ gives the inequalities people in combinatorics want

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How can we generate "special" sops for I_Δ?

Taking a page from the geometry book: failure "for a reason"

Consider a set of points $X = \{X_1, \ldots, X_s\} \subset \mathbb{P}^n$ and integers $\alpha_1, \ldots, \alpha_s > 0$. Many classical problems in algebraic geometry ask about properties of the hypersurfaces that pass through X_i with multiplicity α_i for all i.

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They argue that the expected dimension of this space should be $\max(0, \dim(I_X)_d - \binom{m+n-1}{n})$. When the actual dimension $\dim(I_X \cap I_P^m)_d$ of this space is strictly larger, they say X admits an unexpected hypersurface of degree d

Unexpected hypersurfaces and Macaulay duality

Due to a result of Emsalem and Iarrobino we know there is a relationship between

unexpected hypersurfaces and multiplication maps in $R/(\ell_1^{a_1}, \ldots, \ell_s^{a_s})$

Theorem (Cook-Harbourne-Migliore-Nagel, 2018)

Let $Z = \{X_1, \ldots, X_s\} \subset \mathbb{P}^2$ and $\{L_1, \ldots, L_s\}$ the dual lines. Then Z has an unexpected curve of degree $j+1$ if and only if

$$\times \ell^2 : (R/(L_1^{j+1}, \ldots, L_s^{j+1}))_{j-1} \to (R/(L_1^{j+1}, \ldots, L_s^{j+1}))_{j+1}$$

does not have full rank for general ℓ

What is an inverse system and where does it show up

Consider $R = \mathbb{K}[x_1, \ldots, x_n]$ and $S = \mathbb{K}[y_1, \ldots, y_n]$. S is an R-mod via contraction:

$$x_1^{a_1} \ldots x_n^{a_n} \circ y_1^{b_1} \ldots y_n^{b_n} = \begin{cases} y_1^{b_1 - a_1} \ldots y_n^{b_n - a_n} & \text{if} \quad b_i \geq a_i \quad \text{for all} \quad i \\ 0 & \text{otherwise} \end{cases}$$

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- If I^{-1} is cyclic, R/I is artinian and Gorenstein.
- $\dim I^{-1}_{-j} = \dim(R/I)_j$

When $I^{-1} = (F)$ we call F the Macaulay dual generator of I

Where does it show up

1. The inverse systems of artinian reductions of (CM) squarefree monomial ideals can be interpreted as "stresses" in rigidity theory
2. Emsalem and Iarrobino's result!
3. many more

A way to think of this framework

1: Find an algebra of a specific form failing WLP/SLP

2: Use inverse systems to translate the failure to an interesting geometric property

Failure of WLP/SLP implying interesting geometric behavior is very common: Togliatti systems, Perazzo algebras, etc

When life gives you an algebra failing the WLP...

Consider a squarefree monomial ideal $I \subset \mathbb{K}[x_1, \ldots, x_n]$ and $J = I + (x_1^a, \ldots, x_n^a)$. It is known that WLP/SLP for monomial ideals can be checked by taking $\ell = x_1 + \cdots + x_n$.

Our framework is:

1. Find $A = R/J$ failing the WLP for special J (and I)
2. Apply inverse systems (somehow!) to translate failure to an interesting combinatorial property

... make a regular sequence!

Let I be a Gorenstein squarefree monomial ideal and $J = I + (x_1^a, \dots, x_n^a)$. Assume $A = R/J$ fails the WLP due to surjectivity in degree $d-1 \to d$

The idea

If $\times\ell$ is not surjective, $\times\ell^T$ is not injective. A polynomial $F \in \ker \times\ell^T$ should be the Macaulay dual generator of $I + \theta$, where θ is a system of parameters of I

We call a system of parameters arising this way an unexpected system of parameters of I (or Δ)

A probabilistic approach to failure: expecting the unexpected

Theorem (-, 2025+)

Let I be a Gorenstein squarefree monomial ideal of codimension $\leq n-2$. Then the elementary symmetric polynomials form an unexpected system of parameters of I. In particular,*

$$\frac{R}{I + (x_1^{d+2}, \ldots, x_n^{d+2})}$$

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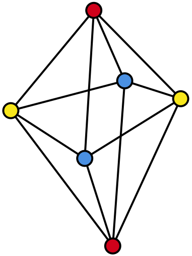
Corollary (Miró-Roig, Migliore, Nagel - 2010)

For every n, the algebra

$$\frac{R}{(x_1 \dots x_n, x_1^n, \dots, x_n^n)}$$

fails the WLP. (Take $I = (x_1 \dots x_n)$)

A famous class of simplicial complexes in combinatorics is the class of balanced complexes: these are complexes that can be colored in such a way that every maximal face has exactly one vertex of each color.



A combinatorial meaning to linear unexpectedness

A famous class of simplicial complexes in combinatorics is the class of balanced complexes: these are complexes that can be colored in such a way that every maximal face has exactly one vertex of each color.

Theorem (-, 2025+)

If a simplicial complex is homeomorphic to a sphere (stronger than Gorenstein!), it admits an unexpected lsop if and only if it is balanced*

Proof: uses a combinatorial result that relies on the fundamental group

What is the intuition for unexpected sops?

One way to think of a sop θ of $I \subset R$ being unexpected, is if the following two conditions are satisfied:

1. The socle degree of $R/(I, \theta)$ is d
2. $x_i^a \in I + \theta$ for some $a \ll d$ and every i

From this perspective, we get to the following question that is interesting on its own

Question

Given a squarefree monomial ideal $I \subset R$ and a pair of integers (d, a), when is there a sop θ of I such that

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1. socle degree of $R/(I, \theta)$ is d
2. $x_i^a \in I + \theta$ for every i
3. Bonus: $x_1 + \cdots + x_n \in I + \theta$

General sops are really bad to guarantee condition 2 for pairs $a \ll d$

The WLP comes into play when we add condition 3

Rees algebras: Rank of a matrix and analytic spread

In the theory of Rees algebras: computing analytic spread = computing ranks of matrices. Our results can be stated in terms of analytic spread:

a certain ideal has maximal analytic spread $\implies$ no unexpected sop

unexpected sop $\implies$ a certain ideal doesn't have maximal analytic spread

Two seemingly distinct questions

Let F be a polynomial of degree d in n variables and $\nabla \cdot F = F_{x_1} + \cdots + F_{x_n}$

Question 1

For which pairs d, n is there a solution F of $\nabla \cdot F = 0$ such that no monomial in F is divisible by x_i^a?

Question 2

Let $A = R/I$ be an artinian Gorenstein algebra such that $x_i^a \in I$ for every i and $x_1 + \cdots + x_n \in I$. What is the highest possible socle degree of A?

What if we add "simplicial restrictions" to the questions above?

The preprint