Lefschetz properties and coinvariant stresses

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July 22, 2025

The beginning

A **simplicial complex** Δ is a collection of subsets (called faces) of $[n]$ such that

$$\tau \subset \sigma \in \Delta \implies \tau \in \Delta.$$

Every simplicial complex can be realized as a topological space in some $\mathbb{R}^n$

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One of the first results in combinatorial commutative algebra due to Stanley says if $\Delta \cong S^d$, then the **Stanley-Reisner ideal**

$$I_\Delta = (\prod_{v \in \sigma} x_v \colon \sigma \notin \Delta) \subset \mathbb{K}[x_v \colon v \text{ a vertex of } \Delta] \text{ is Gorenstein}$$

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The g-conjecture (originally proposed by McMullen in 1971)

A polynomial $h(t)$ with integer coefficients is the numerator of the Hilbert series of R/I where I is a Gorenstein squarefree monomial ideal if and only if

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3. $h(t)$ is symmetric (palindromic) (Gorensteiness)
4. $h(t) = \sum_{i=0}^{d} h_i t^i$, where the sequence $h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{\lfloor \frac{d}{2} \rfloor} - h_{\lfloor \frac{d}{2} \rfloor - 1}$ is the Hilbert series of an artinian algebra ???????

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It turns out that (4) can be shown by finding a linear system of parameters θ of R/I and a linear form ℓ such that

$$\left[\frac{R}{I+\theta}\right]_j \xrightarrow{\times \ell} \left[\frac{R}{I+\theta}\right]_{j+1} \text{ has full rank for every } j$$

Condition (4) and its different interpretations/implications

1. (Geometry/topology) Condition (4) holds if $h(t)$ is the Hilbert series of the cohomology ring of a smooth irreducible complex projective variety (Hard Lefschetz theorem)

2. (Combinatorics) Condition (4) can be used to show sequences are unimodal

3. (Algebra) Condition (4) can be seen as a definition of "regular elements for artinian rings"

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An artinian algebra $A = R/I$ is said to satisfy the *Weak Lefschetz property* (WLP) if there exists a linear form L such that the multiplication maps by L have full rank in every degree. For monomial ideals, it suffices to consider $L =$ sum of variables

Everything should have the WLP!

Theorem (Stanley, 1980)

Artinian monomial complete intersections satisfy the strong Lefschetz property and in particular, the WLP.

Theorem (Stanley, 1980)

If Δ is a simplicial polytope, there exists a linear system of parameters of R/I_Δ that produces an algebra satisfying the WLP

Theorem (Borel?)

The coinvariant ring of the symmetric group $\mathbb{K}[x_1, \ldots, x_n]/(e_1, \ldots, e_n)$ where $e_i =$ sum of squarefree monomials of degree i satisfies the WLP.

Almost everything should have the WLP!

In 2007, Brenner and Kaid used stability of syzygy bundles to show (as a corollary) that the algebra

$$\frac{\mathbb{K}[x,y,z]}{(x^3,y^3,z^3,xyz)} \text{ fails the WLP}$$

In 2011, Migliore, Miró-Roig and Nagel extended this result using Liaison theory and resolutions:

Theorem (MNS, 2011)

For any $n > 2$, the algebra

$$\frac{\mathbb{K}[x_1,\ldots,x_n]}{(x_1^n,\ldots,x_n^n,x_1\ldots x_n)} \text{ fails the WLP.}$$

Algebras failing WLP/SLP are often viewed as rare and failure "happens for a reason"

"Good" monomial algebras... fail? the WLP: a Stanley-Reisner perspective

$$\underbrace{x_1 \dots x_n}_{\text{squarefree part (a complete intersection)}} + \underbrace{(x_1^n, \dots, x_n^n)}_{\text{pure powers}}$$

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Theorem (-, 2025+)

Let $I \subset R$ be a Gorenstein squarefree monomial ideal where every variable appears in a generator of I and let $d+1$ be the Krull dimension of R/I. Then

$$\frac{R}{I + (x_1^{d+2}, \dots, x_n^{d+2})} \text{ fails the WLP.}$$

Failure happens at the same spot and due to the same reason as in the previous results

"Good" monomial algebras... fail? the WLP: what goes in the proof

A naive strategy for proving failure of WLP is to find an element in the kernel of some multiplication map (or its transpose), and prove that the Hilbert series is increasing (or decreasing) at the spot.

1. The inequality of the Hilbert series can be shown using basic double links from liaison theory

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A naive strategy for proving failure of WLP is to find an element in the kernel of some multiplication map (or its transpose), and prove that the Hilbert series is increasing (or decreasing) at the spot.

1. The inequality of the Hilbert series can be shown using basic double links from liaison theory
2. To find the correct element in the kernel, we need polynomial systems of PDEs! (a.k.a Matlis/Macaulay duality in CA or harmonics in AC)

A brief pause: Macaulay duality and systems of PDEs

Given two polynomial rings $R = \mathbb{K}[x_1, \ldots, x_n], S = \mathbb{K}[y_1, \ldots, y_n]$ where $\mathbb{K}$ is a field of char 0 and a monomial $x_1^{a_1} \ldots x_n^{a_n}$ of degree t, define:

$$x_1^{a_1} \ldots x_n^{a_n} \circ F(y_1, \ldots, y_n) = \frac{\partial^t F}{\partial y_1^{a_1} \ldots y_n^{a_n}}$$

Theorem (Macaulay duality)

There is a bijection (up to multiplication by scalars) of Artinian Gorenstein algebras of the form R/I and homogeneous polynomials $F \in S$ given by

$$F \mapsto R/\operatorname{Ann}(F),$$

where $\operatorname{Ann}(F) = \{g \in R : g \circ F = 0\}$ *is an ideal of* R.

Simplicial complexes meet symmetric functions

It is known that for any Gorenstein squarefree monomial ideal I, the ideal $I + (e_1, \ldots, e_n)$ is artinian Gorenstein.

Theorem (-, 2025+)

Let Δ be a d-sphere with orientation $\varepsilon_1 F_1 + \cdots + \varepsilon_s F_s \in \tilde{H}_d(\Delta; \mathbb{K})$. Set

$$V_\Delta = \varepsilon_1 x_{F_1} V(F_1) + \cdots + \varepsilon_s x_{F_s} V(F_s),$$

where $x_{F_i} = \prod_{v \in F_i} x_v$ and $V(F_i) = \prod_{j<k \in F_i} (x_j - x_k)$ is the Vandermonde determinant on F_i. Then

$$\mathrm{Ann}(V_\Delta) = I_\Delta + (e_1, \ldots, e_n).$$

Should the WLP be expected at all?

Theorem (-, 2025+)

Let Δ be a d dimensional simplicial complex such that $d > 0$, $H_d(\Delta; \mathbb{K}) \neq 0$ and $f_{d-1} \geq f_d$. Then

$$\frac{\mathbb{K}[x_1, \ldots, x_n]}{I + (x_1^{d+2}, \ldots, x_n^{d+2})} \text{ fails the WLP.}$$

Failure is caused by the polynomial V_Δ being in the kernel of the transpose of a multiplication map that should be injective

"Everything" fails the WLP

Consider a generalized Erdős-Rényi model to generate random complexes of dimension d starting from a full $(d-1)$–complex on n vertices, and let $\frac{c}{n}$ be the "coin flip" parameter.

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Theorem (-, 2025+)

For every $d > 0$ there exists an integer $c_d < d+1$ such that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\mathbb{K}[x_1,\ldots,x_n]}{I_\Delta + (x_1^{d+2},\ldots,x_n^{d+2})} \textbf{ fails the WLP} \right) = 1,$$

where $c \in (c_d, d+1)$.

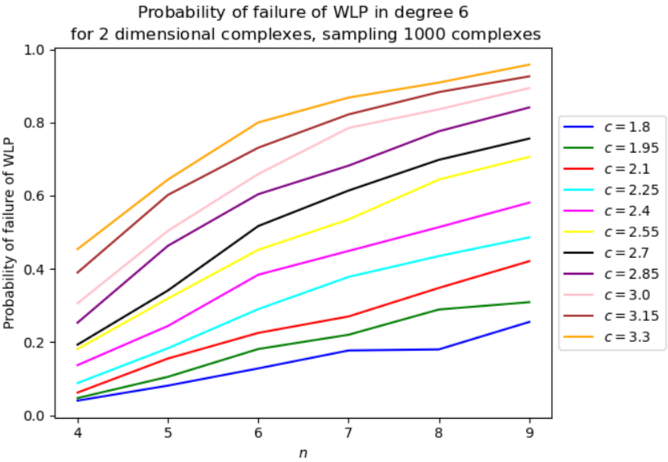
$c_1 = 1$ and c_d can be computed numerically for higher d by solving a simple optimization problem

Maybe not "everything"?

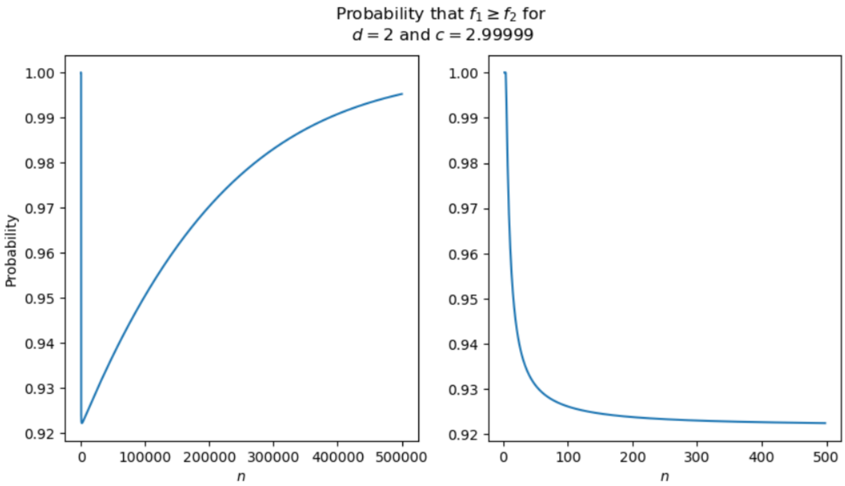
It turns out that c_d converges really quickly to $d+1$. In particular, the size of the interval goes to 0 extremely fast

d	2	3	4	5	6	7	8
c_d	2.783	3.91	4.962	5.984	6.993	7.997	8.998

In this setting probably no



Computations can be misleading



Further directions and applications: when life gives you a monomial algebra failing the WLP

For Gorenstein ideals I_Δ, failure of WLP happened because of the *existence* of a special polynomial V_Δ that is the Macaulay dual generator of $I_\Delta + \theta$, where θ is a (nonlinear) sop of I_Δ. This is not a coincidence. In general it seems to be that:

Existence of "unexpected" sop $\longleftrightarrow$ Monomial algebras failing WLP

And in particular, we can find "special" sops of monomial ideals by studying failure of WLP

Further directions and applications: the algebraic g-conjecture for nongeneral sops

Coinvariant algebraic g-conjecture

Let Δ be a simplicial complex homeomorphic to a d-dimensional sphere. does the ring

$$\frac{\mathbb{K}[x_1, \ldots, x_n]}{I_\Delta + (e_1, \ldots, e_{d+1})}$$

satisfy the SLP? If Δ is the boundary of a simplicial polytope, does the ring satisfy the Hodge-Riemann relations?

For $d = 1, 2$ yes*! (joint ongoing work with Mitsuki Hanada)

The preprint