Gluing failures to achieve success: simplicial complete intersections

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The role of algebra, geometry and combinatorics in the study of WLP

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If the motto in geometry is that "algebras fail the WLP for a reason", then what is the combinatorial meaning of failure of WLP?

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we will focus on the second question

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$$f = \prod_{i<j}(x_i - x_j)(m_{(4,2)} - m_{(3,3)} + m_{(2,2,2)})$$

f is a product of the Vandermonde determinant V and a symmetric polynomial, so it is anti-symmetric

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Inverse systems

We can view S as an R-module by contraction, for a monomial $t \in S$:

$$x_i \circ t = \begin{cases} \dfrac{t}{y_i} & \text{if} \quad y_i | t \\ 0 & \text{otherwise} \end{cases}$$

The annihilator of a homogeneous polynomial F in S is $F^{-1} := \mathrm{Ann}(F) = \{g \in R : g \circ F = 0\}$, which is an artinian Gorenstein ideal of R

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In our case:

$$f^{-1} = (x_1 + x_2 + x_3 + x_4, x_1x_2x_3x_4, h_5, h_6) = (e_4) + (e_1, h_5, h_6)$$

A (symmetric!) complete intersection!

The wonderful case of the symmetric polynomial that is also a squarefree monomial

The symmetric polynomial $e_n = x_1 \dots x_n \in \mathbb{K}[x_1, \dots, x_n]$ can also be viewed as the Stanley-Reisner ideal of the boundary of a simplex, which is the simplest case of a simplicial complex with top homology.

e_1, h_5, h_6 is a system of parameters (s.o.p.) for the Stanley-Reisner ideal (e_4), (which happens to be a complete intersection!).

How can we recreate this process for other squarefree monomial complete intersections? (i.e. more complicated triangulations of spheres)

Putting on the geometer hat: homological extensions of anti-symmetric polynomials

Let Δ be a pure d-dimensional simplicial complex on $[n]$ with maximal faces $\sigma_1, \ldots, \sigma_s$. Let $f \in B = \mathbb{K}[z_1, \ldots, z_{d+1}]$ be an antisymmetric polynomial. Let $R = \mathbb{K}[x_1, \ldots, x_n]$ and

$$\varphi_i : B \hookrightarrow R \quad \varphi_i(z_j) = x_{\ell_j} \quad \text{where} \quad \sigma_i = \{\ell_1, \ldots, \ell_{d+1}\}$$

Assume $H_d(\Delta; \mathbb{K}) \neq 0$. For a nonzero homology class $\varepsilon = \varepsilon_1\sigma_1 + \cdots + \varepsilon_s\sigma_s$, define

$$F_\varepsilon = \varepsilon_1 \prod_{k \in \sigma_1} x_k \varphi_1(f) + \cdots + \varepsilon_s \prod_{k \in \sigma_s} x_k \varphi_s(f)$$

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Moreover, F_ε is the dual of the CI $(x_1x_2, x_3x_4, x_5x_6, e_1, h_5, h_6)$ of codimension 6

Simplicial complete intersections

A sequence $g_1, g_2, \ldots, g_s$ of symmetric polynomials is a simplicial CI if

1. $(e_{s+1}) + (g_1, \ldots, g_s)$ is an artinian CI
2. $e_1 \in (g_1, \ldots, g_s)$
3. $I_\Delta + (g_1, \ldots, g_s)$ is artinian for every $(s-1)$-dimensional complex Δ
4. There exists an anti-symmetric polynomial f such that for every Gorenstein* complex Δ

$$(I_\Delta + (g_1, \ldots, g_s)) = F_\varepsilon^{-1}$$

where ε is an orientation of Δ

$(e_1, \ldots, e_{n-1})$ is a simplicial CI for every n

Some upshots and future questions

Let $(d, a) \in \{(2,4), (2,8), (2,12), (3,5), (3,7), (3,8), (3,9)\}$

Theorem (Probabilistic failure)

There exists an integer $c_d < d+1$ such that if $c_d < c < d+1$, then under the Linial-Meshulam model for d-complexes with parameter $\frac{c}{n}$:

$$\lim_{n\to\infty} \mathbb{P}\left(\Delta\colon \frac{\mathbb{K}[x_1,\ldots,x_n]}{I_\Delta + (x_1^a,\ldots,x_n^a)} \quad \textit{fails the WLP}\right) = 1$$

Proof involves checking some anti-symmetric polynomial is the dual of a simplicial CI

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2. Failure of WLP of $I_\Delta + (x_1^{a_1}, \dots, x_n^{a_n})$ due to surjectivity can be used to find polynomials whose duals are CIs

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5. (unexpected g-conjecture) Do homological extensions define algebras with SLP?