# Inverse systems, Lefschetz properties and random complexes

Thiago Holleben

Dalhousie University

June 7, 2025

# The beginning

# The beginning

In 2003, Migliore and Miró-Roig asked the following question

**Question**

Given an integer $n$, let $A(n)$ denote the maximal number of generators $q$ such that every artinian algebra with at most $q$ generators satisfies the WLP. Does $A(n)$ exist for every $n$?

Since Harima, Migliore, Nagel and Watanabe showed that every complete intersection in 3 variables satisfies the WLP, it was already known that $A(3) \geq 3$ (and in particular $A(3)$ exists)

## The beginning

In 2003, Migliore and Miró-Roig asked the following question

**Question**

Given an integer $n$, let $A(n)$ denote the maximal number of generators $q$ such that every artinian algebra with at most $q$ generators satisfies the WLP. Does $A(n)$ exist for every $n$?

Since Harima, Migliore, Nagel and Watanabe showed that every complete intersection in 3 variables satisfies the WLP, it was already known that $A(3) \geq 3$ (and in particular $A(3)$ exists)

In 2007 Brenner and Kaid used geometric methods to show that the algebra

$$\frac{\mathbb{K}[x, y, z]}{(x^3, y^3, z^3, xyz)}$$

fails the WLP, which means $A(3) = 3$.

# An upper bound for $A(n)$: the original motivation

In 2011, Migliore, Miró-Roig and Nagel showed that monomial *almost complete intersections* failed the WLP very frequently, that is

$$\frac{\mathbb{K}[x_1, \ldots, x_n]}{(x_1^n, \ldots, x_n^n, x_1 \ldots x_n)} \quad \text{fails the WLP for every } n \geq 3$$

and in particular if $A(n)$ exists, it must be $n$

# An upper bound for $A(n)$: the original motivation

In 2011, Migliore, Miró-Roig and Nagel showed that monomial *almost complete intersections* failed the WLP very frequently, that is

$$\frac{\mathbb{K}[x_1,\ldots,x_n]}{(x_1^n,\ldots,x_n^n,x_1\ldots x_n)} \quad \text{fails the WLP for every } n \geq 3$$

and in particular if $A(n)$ exists, it must be $n$

Note that the monomial almost CI above can be seen as

$$\underbrace{(x_1\ldots x_n)}_{\text{squarefree part}} + \underbrace{(x_1^n,\ldots,x_n^n)}_{\text{pure powers}}$$

# A generalization from a Stanley-Reisner perspective

Note that the monomial almost complete intersections from before are always of the form

$$\underbrace{(x_1 \dots x_n)}_{\text{squarefree part}} + \underbrace{(x_1^n, \dots, x_n^n)}_{\text{pure powers}}$$

Our first generalization of this result is the following

**Theorem (-, 2025+)**

*Let $I \subset R = \mathbb{K}[x_1, \dots, x_n]$ be a Gorenstein squarefree monomial ideal such that every variable of $R$ appears in at least one generator of $I$ and* $\dim \frac{R}{I} = d + 1 > 1$. *Then*

$$\frac{R}{I + (x_1^{d+2}, \dots, x_n^{d+2})} \quad \textit{fails the WLP}$$

# A generalization from a Stanley-Reisner perspective

Note that the monomial almost complete intersections from before are always of the form

$$\underbrace{(x_1 \dots x_n)}_{\text{squarefree part}} + \underbrace{(x_1^n, \dots, x_n^n)}_{\text{pure powers}}$$

Our first generalization of this result is the following

**Theorem (-, 2025+)**

*Let $I \subset R = \mathbb{K}[x_1, \dots, x_n]$ be a Gorenstein squarefree monomial ideal such that every variable of $R$ appears in at least one generator of $I$ and* $\dim \frac{R}{I} = d + 1 > 1$. *Then*

$$\frac{R}{I + (x_1^{d+2}, \dots, x_n^{d+2})} \quad \textit{fails the WLP}$$

The result is clearly false if we remove the assumption on the generators since $I = 0$ gives us a monomial CI. The $d > 0$ assumption is to exclude $I = (x_1 x_2)$

# The setup

A **simplicial complex** $\Delta$ is a collection of subsets (called faces) of $[n]$ such that

$$\tau \subset \sigma \in \Delta \implies \tau \in \Delta$$

## Definition

Given a simplicial complex $\Delta$ on $[n]$ vertices, its **Stanley-Reisner** ideal is the ideal

$$I_\Delta = (x_{i_1} \dots x_{i_s} : \{i_1, \dots, i_s\} \notin \Delta)$$

The **dimension** of $\Delta$ is the size of a maximal face of $\Delta$ $-1$,
$\dim \Delta + 1 = \dim \frac{R}{I_\Delta}$

$e_i$ = sum of every squarefree monomial of degree $i$

# From complete intersections to Gorenstein: squarefree monomial ideals

The proof of both results follows a very similar strategy:

1. Find an element in the kernel of the transpose of a multiplication map
   For CIs: Vandermonde determinant, for Gorenstein ideals it is trickier
2. Prove that the Hilbert function at that step is decreasing Both results need basic double links from liaison theory

# From complete intersections to Gorenstein: squarefree monomial ideals

The proof of both results follows a very similar strategy:

1. Find an element in the kernel of the transpose of a multiplication map For CIs: Vandermonde determinant, for Gorenstein ideals it is trickier
2. Prove that the Hilbert function at that step is decreasing Both results need basic double links from liaison theory

The generalization of step 1 is a consequence of the following observation

**The main idea**

Macaulay duality takes multiplication maps to transposes, so that failure of surjectivity is equivalent to some artinian ideal $I + L \subseteq J$ containing a special element in its inverse system

# The universal system of parameters (sop): elementary symmetric polynomials

We use the idea by showing the following

**Theorem (-, 2025+)**

*Let $\Delta$ be a d-dimensional complex such that $\tilde{H}_d(\Delta, \mathbb{K}) \neq 0$, and let $\varepsilon_1 F_1 + \cdots + \varepsilon_s F_s$ be a nonzero element. Then*

$$\varepsilon_1 x_{F_1} V(F_1) + \cdots + \varepsilon_s x_{F_s} V(F_s) \in (I_\Delta + (e_1, \ldots, e_{d+1}))^{-1},$$

*where $x_{F_i} = \prod_{j \in F_i} x_j$, $V(F_k) = \prod_{i<j \in F_k} (x_i - x_j)$ and $\mathbb{K}$ is the base field*

The polynomial above has degree $\binom{d+2}{2}$ and is what ends up causing failure of WLP

Note that if $I_\Delta$ is a CI, we end up generating infinite families of polynomials that are the dual generators of CIs

# Generalizing step 2: using basic double links to prove "small miracles"

Let $A_\Delta = \frac{R}{I_\Delta + (x_1^{d+2}, \ldots, x_n^{d+2})}$ and $f_i =$ number of faces of $\Delta$ of dimension $i$.
Then the difference $HF(A_\Delta, \binom{d+2}{2} - 1) - HF(A_\Delta, \binom{d+2}{2})$ is equal to (for low $d$)

1. $d = 8 : 4f + 971f_5 + 21609f_6 + 151936(f_7 - f_8)$
2. $d = 7 : 56f_4 + 1624f_5 + 11096(f_6 - f_7)$
3. $d = 6 : 3f_3 + 145f_4 + 981(f_5 - f_6)$
4. $d = 5 : 15f_3 + 111(f_4 - f_5)$
5. $d = 4 : 2f_2 + 16(f_3 - f_4)$
6. $d = 3 : 4(f_2 - f_3)$
7. $d = 2 : f_1 - f_2$
8. $d = 1 : 0$

The inequality we want only depends on the sign of $f_{d-1} - f_d$!

# Using basic double links to spot numerical coincidences

It turns out that the reason there is such a nice simplification to the problem in this setting is because of the following numerical coincidence (shown below for $d = 4$) that can be proven using basic double links

$$HS(R/I_1, T) = ... + 365\, T^9 + 381\, T^{10} + 365\, T^{11} + ...$$
$$HS(R/I_2, T) = ... + 80\, T^9 + 68\, T^{10} + 52\, T^{11} + ...$$

where $I_1 = (x_1^5, \dots, x_5^5), I_2 = (x_1, x_2^5, \dots, x_5^5) \subset \mathbb{K}[x_1, \dots, x_5]$

# The full generalization

The first main result is the following

**Theorem (-, 2025+)**

*Let* $d > 0$ *and* $\Delta$ *a* $d$*-dimensional complex such that* $f_{d-1} \geq f_d$ *and* $\tilde{H}_d(\Delta; \mathbb{K}) \neq 0$. *Then*

$$\frac{\mathbb{K}[x_1, \ldots, x_n]}{I_\Delta + (x_1^{d+2}, \ldots, x_n^{d+2})}$$

*fails the WLP due to surjectivity.*

# Should the WLP of monomial ideals really be expected?

Consider a generalized Erdős-Rényi model for $d$-dimensional complexes on $n$ vertices with "coin flip" parameter $\frac{c}{n}$. We show the following:

**Theorem ("Everything" fails, -, 2025+)**

*For every $d > 0$, there exists an integer $c_d < d + 1$ such that if $c_d < c < d + 1$, then*

$$\lim_{n\to\infty} \mathbb{P}\left( \frac{\mathbb{K}[x_1, \ldots, x_n]}{I_\Delta + (x_1^{d+2}, \ldots, x_n^{d+2})} \quad \textit{fails the WLP} \right) = 1$$

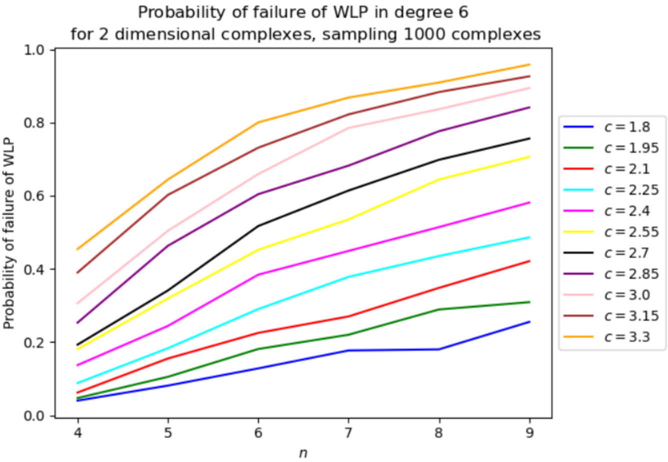
$c_1 = 1$ and for higher $d$, $c_d$ can be numerically computed by solving an optimization problem

# Maybe yes?

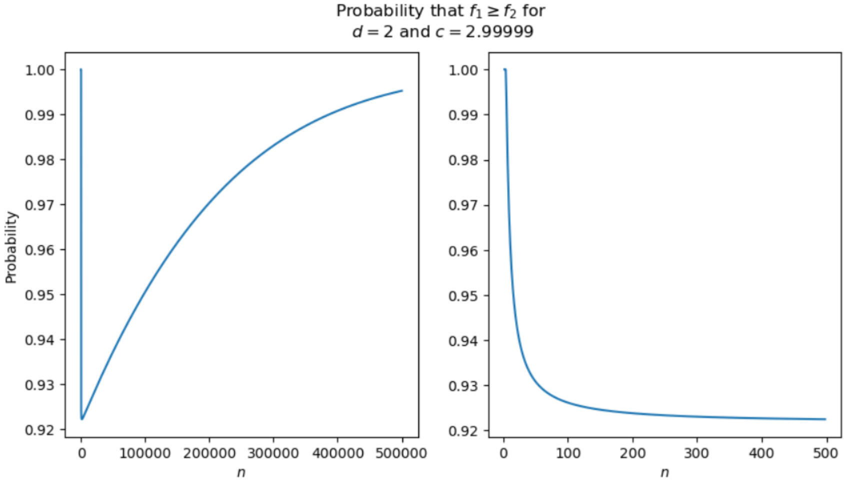
It turns out that $c_d$ converges really quickly to $d+1$. In particular, the size of the interval goes to 0 extremely fast

| $d$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| $c_d$ | 2.783 | 3.91 | 4.962 | 5.984 | 6.993 | 7.997 | 8.998 |

# In this setting probably no



# Computations can be misleading



# Further directions and applications

1. Perazzo algebras failing WLP (Nagata idealization)
2. Find "interesting" systems of parameters of squarefree monomial ideals

## Coinvariant algebraic $g$-conjecture

Let $\Delta$ be a simplicial complex homeomorphic to a $d$-dimensional sphere. does the ring

$$\frac{\mathbb{K}[x_1, \ldots, x_n]}{I_\Delta + (e_1, \ldots, e_{d+1})}$$

satisfy the SLP? If $\Delta$ is the boundary of a simplicial polytope, does the ring satisfy the Hodge-Riemann relations?

# Further directions and applications

1. Perazzo algebras failing WLP (Nagata idealization)
2. Find "interesting" systems of parameters of squarefree monomial ideals

### Coinvariant algebraic $g$-conjecture

Let $\Delta$ be a simplicial complex homeomorphic to a $d$-dimensional sphere. does the ring

$$\frac{\mathbb{K}[x_1, \ldots, x_n]}{I_\Delta + (e_1, \ldots, e_{d+1})}$$

satisfy the SLP? If $\Delta$ is the boundary of a simplicial polytope, does the ring satisfy the Hodge-Riemann relations?

For $d = 1$, yes! (joint ongoing work with Mitsuki Hanada)