Lefschetz properties of squarefree monomial ideals via Rees algebras

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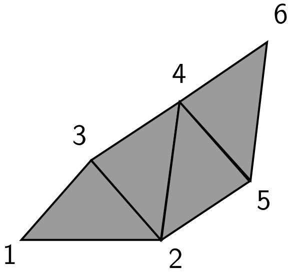
June 28

Stanley-Reisner, Facet (and incidence) ideals

A simplicial complex Δ on vertex set $[n]$ is a collection of subsets Δ of $[n]$ such that $\tau \subset \sigma \in \Delta \implies \tau \in \Delta$. We write $\Delta = \langle F_1, \ldots, F_s \rangle$ if $F_1, \ldots, F_s$ are the facets (maximal subsets) of Δ.



$$\Delta = \langle \{1, 2, 3\}, \{2, 3, 4\}, \{2, 4, 5\}, \{5, 4, 6\} \rangle$$

Stanley-Reisner, Facet (and incidence) ideals

Let $S = k[x_1, \ldots, x_n]$ and $\Delta = \langle F_1, \ldots, F_s \rangle$ a simplicial complex with vertex set $[n]$.

- The **Stanley-Reisner** ideal of Δ is the ideal

$$\mathcal{N}(\Delta) = (\prod_{i \in B} x_i : B \notin \Delta) \subset S$$

- The **Facet** ideal of Δ is the ideal

$$\mathcal{F}(\Delta) = (\prod_{i \in F_1} x_i, \ldots, \prod_{i \in F_s} x_i) \subset S$$

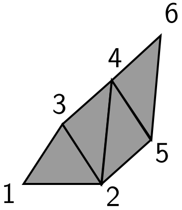
Both constructions give bijections between simplicial complexes and squarefree monomial ideals

Stanley-Reisner, Facet (and incidence) ideals

$$\mathcal{N}(\Delta) = (\prod_{i \in B} x_i : B \notin \Delta), \quad \mathcal{F}(\Delta) = (\prod_{i \in F_1} x_i, \dots, \prod_{i \in F_s} x_i)$$

$(x_1x_4, x_1x_5, x_3x_5, x_1x_6, x_2x_6, x_3x_6)$ ⟷

$\mathcal{N}(\Delta)$



Δ

↕

$(x_1x_2x_3, x_2x_3x_4, x_2x_4x_5, x_4x_5x_6)$

$\mathcal{F}(\Delta)$

Lefschetz properties

Let I be a monomial ideal of $S = k[x_1, \dots, x_n]$ such that $A = S/I$ is artinian.

Definition

We say A satisfies the **weak Lefschetz property (WLP)** if the multiplication maps

$$\times L : A_i \to A_{i+1}$$

by some linear form $L \in S_1$ have full rank for every i.
If moreover the maps

$$\times L^j : A_i \to A_{i+j}$$

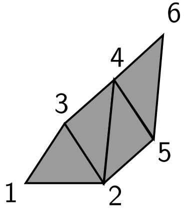
have full rank for every i, j, we say A satisfies the **strong Lefschetz property (SLP)**

Since I is monomial, we can take $L = x_1 + \cdots + x_n \in S_1$

An example with the SLP

$(x_1x_4, x_1x_5, x_3x_5, x_1x_6, x_2x_6, x_3x_6)$ ⟷

$\mathcal{N}(\Delta)$



Δ

The algebra

$$A(\Delta) = k[x_1, \ldots, x_6]/(\mathcal{N}(\Delta), x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2)$$

has the SLP whenever char $k \neq 2$

An example with the SLP

Let $\mathcal{N}(\Delta) = (x_1x_4, x_1x_5, x_3x_5, x_1x_6, x_2x_6, x_3x_6) \subset S = k[x_1, \dots, x_6]$, Then

$$A(\Delta) = \frac{S}{(\mathcal{N}(\Delta), x_1^2, \dots, x_6^2)}$$

and

$$A(\Delta)_1 \xrightarrow{\begin{array}{c c} & \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{array}{c} x_1x_2 \\ x_1x_3 \\ x_2x_3 \\ x_2x_4 \\ x_2x_5 \\ x_3x_4 \\ x_4x_5 \\ x_4x_6 \\ x_5x_6 \end{array} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}} A(\Delta)_2$$

has full rank in every odd characteristic

Incidence matrices (everywhere!)

The two matrices that represent the maps we just saw have very particular structures:

$$A(\Delta)_1 \xrightarrow{\begin{array}{c|cccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \hline x_1x_2 & 1 & 1 & 0 & 0 & 0 & 0 \\ x_1x_3 & 1 & 0 & 1 & 0 & 0 & 0 \\ x_2x_3 & 0 & 1 & 1 & 0 & 0 & 0 \\ x_2x_4 & 0 & 1 & 0 & 1 & 0 & 0 \\ x_2x_5 & 0 & 1 & 0 & 0 & 1 & 0 \\ x_3x_4 & 0 & 0 & 1 & 1 & 0 & 0 \\ x_4x_5 & 0 & 0 & 0 & 1 & 1 & 0 \\ x_4x_6 & 0 & 0 & 0 & 1 & 0 & 1 \\ x_5x_6 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}} A(\Delta)_2$$

Taking rows as exponents we have the ideal

$$(x_1x_2, x_1x_3, x_2x_3, x_2x_4, x_2x_5, x_3x_4, x_4x_5, x_4x_6, x_5x_6)$$

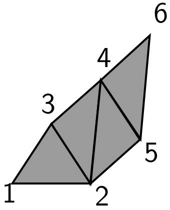
Incidence matrices (everywhere!)

We call the matrices that represent the multiplication by L maps in $A(\Delta)$ the **incidence matrices** of Δ.
Taking rows of an incidence matrix as exponents we have an **incidence ideal** of Δ. Incidence ideals are ideals in the **incidence ring** of Δ:

$$S_{\Delta} = \mathbb{C}[x_{\tau} : \tau \in \Delta]$$

Incidence matrices (everywhere)

- $\times L : A(\Delta)_1 \to A(\Delta)_2$ corresponds to the ideal $(x_1x_2, x_1x_3, x_2x_4, x_2x_4, x_2x_5, x_3x_4, x_4x_5, x_4x_6, x_5x_6)$
- $\times L : A(\Delta)_2 \to A(\Delta)_3$ corresponds to the ideal $(x_{\{1,2\}}x_{\{1,3\}}x_{\{2,3\}}, x_{\{2,3\}}x_{\{2,4\}}x_{\{3,4\}}, x_{\{2,4\}}x_{\{2,5\}}x_{\{4,5\}}, x_{\{4,5\}}x_{\{4,6\}}x_{\{5,6\}})$



Δ

The bipartite property in Combinatorial Commutative Algebra

Let $I(G) = (x_i x_j \colon ij \text{ is an edge of } G)$ be the edge ideal of G

$$
\begin{aligned}
\text{Not bipartite} &\iff \text{The rational map defined by } I(G) \text{ is birational} \\
&\iff I(G) \text{ is of linear type} \\
&\iff I(G)^{(m)} \neq I(G)^m \text{ for some } m \\
&\iff \text{Incidence matrix has full rank (one multiplication map)}
\end{aligned}
$$

But what can we say for simplicial complexes in general?

$\mathcal{F}(\Delta)$ Rees $\implies$ $\mathcal{N}(\Delta)$ Lefschetz

Theorem (-, 2024)

If Δ is connected and pure of dimension 2, then:

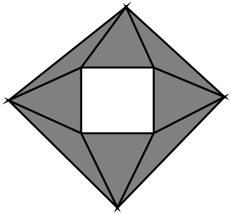
$$\mathcal{F}(\Delta) \text{ is of linear type} \implies A(\Delta) \text{ has the SLP}$$

Which properties of the Rees algebra

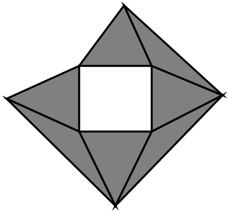
$$S[\mathcal{F}(\Delta)t] = \bigoplus_{i \in \mathbb{N}} t^i \mathcal{F}(\Delta)^i$$

of $\mathcal{F}(\Delta)$ can be translated into information on the Lefschetz properties of $\mathcal{N}(\Delta)$?

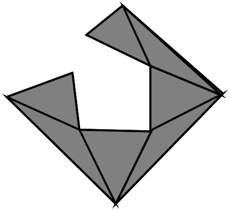
From linear type to Lefschetz properties: sufficient conditions visualized



Linear type results can't be used



Linear type results imply WLP in every odd characteristic



SLP in every odd characteristc

Symbolic powers

Symbolic powers of squarefree monomial ideals

Let $\mathcal{F}(\Delta) \subset S = k[x_1, \ldots, x_n]$ be a squarefree monomial ideal. The m-th symbolic power of $\mathcal{F}(\Delta)$ is:

$$\mathcal{F}(\Delta)^{(m)} = \bigcap_{P \in \mathrm{Ass}(\mathcal{F}(\Delta))} P^m$$

If $\mathcal{F}(\Delta) = (x_1x_2, x_2x_3, x_1x_3)$, then

$$\mathcal{F}(\Delta)^{(2)} = (x_1x_2x_3, x_1^2x_2^2, x_2^2x_3^2, x_1^2x_3^2) \neq \mathcal{F}(\Delta)^2$$

Symbolic powers and Lefschetz properties of squarefree monomial ideals are not compatible

Theorem (-, 2024)

Let Δ be a pure simplicial complex with at least as many facets as vertices.

- *If $\mathcal{F}(\Delta)^{(m)} = \mathcal{F}(\Delta)^m$ for all m, then $A(\Delta)$ fails the SLP due to the largest map not being injective.*

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Corollary (-, 2024)

Let G be a bipartite graph with $n \geq 5$ vertices and $w(G)$ the whiskered graph. Let

$$I(w(G)) = (x_{i_{1,1}}, \ldots, x_{i_{1,n}}) \bigcap \cdots \bigcap (x_{i_{r,1}}, \ldots, x_{i_{r,n}})$$

and $\Delta = \langle \{i_{1,1}, \ldots, i_{1,n}\}, \ldots, \{i_{r,1}, \ldots, i_{r,n}\} \rangle$. Then $A(\Delta)$ fails the SLP.

Lefschetz properties of $A(\Delta)$ versus Rees algebras of $\mathcal{F}(\Delta)$

- Mixed multiplicities $\iff$ failure (or not) of WLP/SLP in characteristic zero, bounds in positive characteristic
- Linear type + low dimension $\implies$ WLP/SLP in characteristic zero
- Symbolic powers = ordinary powers $\implies$ failure of SLP in characteristic zero

From Lefschetz to Rees: Simplicial (mixed) Eulerian numbers

The Eulerian number $A(n,k)$ is the number of permutations of $[n]$ with k ascents

Theorem (Laplace, 1886)

$A(n,k)$ is equal to the volume of the convex hull of the set

$$\left\{ \sum_{i\in I} e_i : I \subset [n], \text{ and } |I| = k \right\}$$

Theorem (Postnikov, 2009)

The mixed volumes of the polytopes above (for $1 \leq k < n$) are all positive

From Lefschetz to Rees: Simplicial (mixed) Eulerian numbers

The following is a corollary of known results about Lefschetz properties of monomial ideals

Corollary (-, 2024)

Let Δ be a pure simplicial complex of dimension d. The mixed volumes of the polytopes given by the convex hull of

$$\left\{ \sum_{i \in I} e_i : I \in \Delta, \text{ and } |I| = k \right\}$$

for $1 \leq k < d$ are all positive

Rees and Lefschetz beyond monomial ideals

- Ranks of multiplication maps for algebras $A(\Delta)$ say whether a set of monomials is algebraic dependent or not $\longrightarrow$ Perazzo forms (via Nagata idealization)

Rees and Lefschetz beyond monomial ideals

- Ranks of multiplication maps for algebras $A(\Delta)$ say whether a set of monomials is algebraic dependent or not $\longrightarrow$ Perazzo forms (via Nagata idealization)
- Analytic spread can always be computed by ranks of matrices that show up as multiplication maps