Algebraic properties of extremal ideals: one (squarefree monomial) ideal to rule them all

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A natural question

Let f be an invariant* of ideals. Given a specific family $\mathcal{A}$ of ideals (e.g. all squarefree monomial ideals on q generators), how can we find a sharp bound c_f such that

$$f(I) \leq c_f \quad \text{for all} \quad I \in \mathcal{A}?$$

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We will show that for every q there exists a single squarefree monomial ideal ideal $\mathcal{E}_q$ such that for several invariants f we have

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for example, for every r (and i): $\beta_i(I^r) \leq \beta_i(\mathcal{E}_q^r)$, $\beta_i(I^{(r)}) \leq \beta_i(\mathcal{E}_q^{(r)})$,
$\beta_i(\overline{I^r}) \leq \beta_i(\overline{\mathcal{E}_q^r})$
$\operatorname{astab}(I) \leq \operatorname{astab}(\mathcal{E}_q)$
$\rho(I) \leq \rho(\mathcal{E}_q)$, $\rho_a(I) \leq \rho_a(\mathcal{E}_q)$, $\operatorname{sdefect}(r, I) \leq \operatorname{sdefect}(r, \mathcal{E}_q)$

(Why) should we expect such an ideal to exist?

We can model squarefree monomial ideals combinatorially, so that the only relevant information for the ideal, is how different variables interact in the generators (how "(hyper)edges" intersect). If we are able to define an ideal that contains all the possible interactions, maybe it would have such property.

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Concretely, we want for every subset of generators, a variable that only divides that set of generators

The (sharp) upper bounds will follow since we will be able to map this ideal into any other squarefree monomial ideal on the same number of generators

Extremal ideals: the definition [EFȘS, 24]

1. $S_{[q]} = \mathbb{K}[y_A \colon A \subseteq [q] \quad \text{and} \quad A \neq \emptyset]$ (one variable per subset of $[q] = \{1, \ldots, q\}$)

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3. **The q-Extremal ideal:** $\mathcal{E}_q = (\varepsilon_1, \ldots, \varepsilon_q) \subset S_{[q]}$

The variable y_A divides ε_i if and only if $i \in A$, so we have the property we wanted

$$\begin{aligned}\mathcal{E}_4 = (&y_1 y_{12} y_{13} y_{14} y_{123} y_{124} y_{134} y_{1234},\\ &y_2 y_{12} y_{23} y_{24} y_{123} y_{124} y_{234} y_{1234},\\ &y_3 y_{13} y_{23} y_{34} y_{123} y_{134} y_{234} y_{1234},\\ &y_4 y_{14} y_{24} y_{34} y_{124} y_{134} y_{234} y_{1234})\end{aligned}$$

Extremal ideals: the hidden hero [EFȘS, 24]

It turns out that for every squarefree monomial ideal $I = (m_1, \ldots, m_q) \subset R$ there exists a map

$$\psi_I : S_{[q]} \to R$$

such that $\psi_I(\varepsilon_i) = m_i$, hence

$$\psi_I(\mathcal{E}_q) = I \quad \text{and} \quad \psi_I(\mathcal{E}_q^r) = I^r$$

$I = (x_1 x_5 x_2 x_7, x_3 x_2 x_7, x_3 x_4 x_6) \subset R = \mathbb{K}[x_1, \ldots, x_7]$

$\psi_I(y_{12}) = x_2 x_7$, $\psi_I(y_{23}) = x_3$, $\psi_I(y_1) = x_1 x_5$, $\psi_I(y_3) = x_4 x_6$, $\psi_I(y_A) = 1$ otherwise

Extremal ideals: the original motivation [EFȘS, 24]

The original "bound" obtained from extremal ideals says

Theorem (EFȘS, 24)

Let I be a squarefree monomial ideal minimally generated by q elements. Then

$$\beta_i(I^r) \leq \beta_i(\mathcal{E}_q^r) \quad \textit{for all} \quad i, r$$

leads to several interesting combinatorial questions in topology and discrete geometry

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Different notions of powers: ψ saves the day once again

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Theorem (CDFHMŞ, 26+)

Let I be a squarefree monomial ideal generated by q elements. Then

1. $\psi_I(\mathcal{E}_q^{(r)}) = I^{(r)}$
2. $\psi_I(\overline{\mathcal{E}_q^r}) = \overline{I^r}$

Invariants of symbolic powers and integral closure

Invariants from the containment problem

1. $\rho(I) = \sup\{\frac{s}{r} : I^{(s)} \not\subset I^r\}$ [BH, 2010]
2. $\rho_a(I) = \sup\{\frac{s}{r} : I^{(s)} \not\subset \overline{I^r}\}$ [GHVT, 2013], [DFMS, 2019]

Measuring differences between powers

1. $\mathrm{sdefect}(r, I) = \mu(I^{(r)}/I^r)$ [GGSVT, 2018]
2. $\mathrm{idefect}(r, I) = \mu(\overline{I^r}/I^r)$

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Theorem (CDFHMŞ, 26+)

Let I be a squarefree monomial ideal minimally generated by q elements. Then

$$\operatorname{sdefect}(2, I) \leq 2^q - q - 1 - \binom{q}{2}.$$

Moreover, this bound is sharp.

The wonders of extremal ideals: turning statements into examples

Proposition (CDFHMŞ, 26+)

If $I = (m_1, m_2, m_3)$ is a squarefree monomial ideal then $\overline{I^r} = I^r$ for all r.

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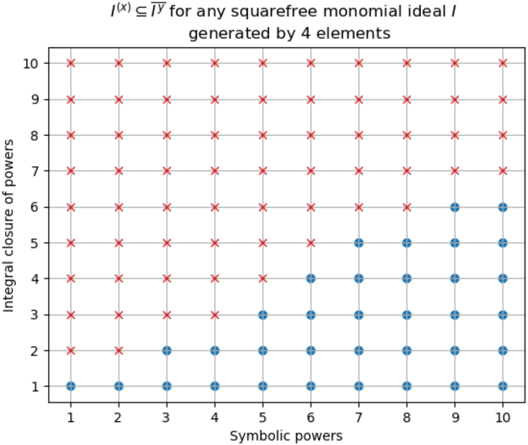

The dangers of extremal ideals: tackling all obstructions at the same time

The q-extremal ideal is defined in a polynomial ring in $2^q - 1$ variables, generated in high degree.

For $q = 5$ most computations are already infeasible

Most of the techniques we use imply there should be an equivalent discrete geometric statement to computing the invariants mentioned so far. This problem might be more tractable, but still very hard

A lot more

There are many more applications of extremal ideals. Other properties that can be studied via extremal ideals are:

associated primes, persistence property, stabilization of associated primes