Lefschetz Properties and Mixed Multiplicities

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Lefschetz Properties

Lefschetz properties have been studied in many different settings including:

- Kähler manifolds
- Sperner properties of posets
- Lozenge tilings
- Laplace equations

Lefschetz Properties

Let $S = k[x_1, \dots, x_n]$ and $I = (x_1^{a_1}, \dots, x_n^{a_n}) + I'$ a monomial ideal such that $A = S/I$ is Artinian.

WLP

We say A has the Weak Lefschetz Property (WLP) in degree i if the multiplication map $\times L : A_i \to A_{i+1}$ by the linear form $L = x_1 + \cdots + x_n$ has full rank.

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For example, if $A = k[a, b, c, d]/(a^2, b^2, c^2, abd)$:

$$A = k \oplus < a, b, c, d > \oplus < ab, ac, ad, bc, bd, cd > \oplus < abc, acd, bcd >$$

$$A_1 \xrightarrow{\begin{array}{c|cccc} & a & b & c & d \\ \hline ab & 1 & 1 & 0 & 0 \\ ac & 1 & 0 & 1 & 0 \\ ad & 1 & 0 & 0 & 1 \\ bc & 0 & 1 & 1 & 0 \\ bd & 0 & 1 & 0 & 1 \\ cd & 0 & 0 & 1 & 1 \end{array}} A_2 \qquad A_2 \xrightarrow{\begin{array}{c|cccccc} & ab & ac & ad & bc & bd & cd \\ \hline abc & 1 & 1 & 0 & 1 & 0 & 0 \\ acd & 0 & 1 & 1 & 0 & 0 & 1 \\ bcd & 0 & 0 & 0 & 1 & 1 & 1 \end{array}} A_3$$

Lefschetz Properties - Matrices

From the definition, we see that A has the WLP if and only if some matrices have full rank.

We can think of the columns/rows of a matrix as either:

- Exponents of monomials (product)
- Coefficients of linear forms (sum)

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Our main goal is to describe the WLP of an algebra in terms of algebraic invariants of such ideals.

A polynomial ring

Let $S = k[x_1, \ldots, x_n]$ and I a monomial ideal such that $A = S/I$ is an Artinian ring.

Definition

We denote by S_I the polynomial ring that has one variable for each nonzero monomial of A:

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Example

Let $I = (x_1^2, \ldots, x_n^2)$. Then

$$S_I = \mathbb{C}[t_m | \; m \text{ squarefree monomial in } n \text{ variables}]$$

WLP in characteristic 0

Let I' be a squarefree monomial ideal of $S = k[x_1, \ldots, x_n]$ and $A = S/I$ an artinian algebra, where $I = (x_1^2, \ldots, x_n^2) + I'$.

The matrices that appear in the study of the WLP of squarefree artinian reductions have very particular properties:

Proposition

Let $A = S/I$ where I is a squarefree monomial ideal plus the squares of the variables in S. Then the matrices that represent the multiplication map by $L = x_1 + \cdots + x_n$ have constant row sum.

Rows as products: an example

Let $S = k[a,b,c,d]$, $I = (a^2, b^2, c^2, d^2, abd)$ and $A = S/I$. The multiplication map $\times L : A_2 \to A_3$ is represented by the following matrix:

$$\begin{array}{c} \\ abc \\ acd \\ bcd \end{array} \begin{array}{c} \begin{array}{cccccc} ab & ac & ad & bc & bd & cd \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

From the rows of the matrix above we define the equigenerated monomial ideal

$$I(2) = (\underbrace{t_{ab}t_{ac}t_{bc}}_{abc}, \underbrace{t_{ac}t_{ad}t_{cd}}_{acd}, \underbrace{t_{bc}t_{bd}t_{cd}}_{bcd})$$

Rank and analytic spread

The following well-known result from Ehrhart theory gives us a connection between rank of matrices and analytic spread:

Theorem

Let M be a matrix such that the sum of the entries in each row is constant equal to d. Then the analytic spread of the monomial ideal defined by the rows (as exponents) is equal to the rank of M.

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Corollary (−, 2023)

This means the WLP of squarefree reductions in characteristic 0 is equivalent to "some" monomial ideals having maximal analytic spread.

The WLP in positive characteristics

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To study the WLP in positive characteristics, we still want to know if some matrix has full rank, but the entries are now taken over some field of nonzero characteristic. A matrix that has full rank in characteristic zero does not necessarily have full rank in positive characteristic:

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

the problem is exactly in the characteristics that divide the nonzero maximal minors

Let $S = k[x_0, \ldots, x_n]$, $\mathfrak{m} = (x_0, \ldots, x_n)$ and J an arbitrary ideal of S. The multigraded Hilbert polynomial of the algebra

$$R(\mathfrak{m}|J) = \bigoplus_{(u,v)\in\mathbb{N}^2} \mathfrak{m}^u J^v / \mathfrak{m}^{u+1} J^v$$

can be written as

$$\sum_{i=0}^{n} e_{(n-i,i)}(\mathfrak{m}|J) u^{n-i} v^i + \text{ terms of lower degree}$$

The nonnegative numbers $e_{(n-i,i)}(\mathfrak{m}|J)$ are called the mixed multiplicities of $\mathfrak{m}$ and J.

Mixed multiplicities, volume and analytic spread

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Using these results we can describe the failure of the WLP in positive characteristics for squarefree reductions in terms of mixed multiplicities.

Mixed multiplicities and the WLP in positive characteristics

Theorem (−, 2023)

Consider the artinian algebra $A = k[x_1, \ldots, x_n]/(I + (x_1^2, \ldots, x_n^2))$ *where* I *is a squarefree monomial ideal. Assume* $\dim A_2 \geq \dim A_1$*. Then* A *either has the WLP in degree* 1 *in every characteristic* $\neq 2$*, or* A *fails the WLP in every characteristic.*

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Moreover, the following are known:

1. WLP in characteristic 2 of squarefree reductions is related to simplicial homology (Migliore, Nagel and Schenck - 2017)

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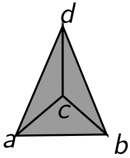
Moreover, the following are known:

1. WLP in characteristic 2 of squarefree reductions is related to simplicial homology (Migliore, Nagel and Schenck - 2017)
2. A criterion for the WLP in degree 1 and characteristic 0 is known in combinatorial terms (Dao and Nair - 2021)

An example

Consider the following simplicial complex Δ



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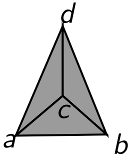

Then $A = k[a,b,c,d]/(a^2,b^2,c^2,d^2,abd)$. To check for WLP in degree 1 we need to check if the following matrix has full rank:

$$A_1 \xrightarrow{\begin{array}{c|cccc} & a & b & c & d \\ ab & 1 & 1 & 0 & 0 \\ ac & 1 & 0 & 1 & 0 \\ ad & 1 & 0 & 0 & 1 \\ bc & 0 & 1 & 1 & 0 \\ bd & 0 & 1 & 0 & 1 \\ cd & 0 & 0 & 1 & 1 \end{array}} A_2$$

$$\begin{cases} (a+b+c+d)a = ab+ac+ad \\ (a+b+c+d)b = ab+bc+bd \\ (a+b+c+d)c = ac+bc+cd \\ (a+b+c+d)d = ad+bd+cd \end{cases}$$

An example

From the matrix we can define the ideal:

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$$I(1) = (t_a t_b, t_a t_c, t_a t_d, t_b t_c, t_b t_d, t_c t_d)$$

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$$I(1) = (t_a t_b, t_a t_c, t_a t_d, t_b t_c, t_b t_d, t_c t_d)$$

The analytic spread of $I(1)$ is 4, in particular the matrix has full rank