Lefschetz Properties and Mixed Multiplicities

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Lefschetz Properties as matrices

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For example if $I = (x^3, y^3, z^3, xy, yz)$ is a monomial ideal of $R = k[x, y, z]$, then $A = R/I$ has the WLP in degree 1 if the following map has full rank:

$$A_1 \xrightarrow{\begin{array}{c} \quad x \quad y \quad z \\ \begin{array}{c} x^2 \\ y^2 \\ z^2 \\ xz \end{array} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \end{array}} A_2$$

First perspective: Hyperplanes

Let $A = k[x_1, \ldots, x_n]/((x_1^{a_1}, \ldots, x_n^{a_n}) + I')$ and

$$\times(x_1 + \cdots + x_n) : A_i \xrightarrow{M} A_{i+1}$$

we can take the entries in the columns of M to be coefficients of linear forms.

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$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccc} a+d & b & c+d \end{array} \\ \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{array} \right) \end{array} \to h = (a+d)(b)(c+d) \in \mathbb{C}[a, b, c, d]$$

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Then A has the WLP in degree 1 and characteristic 0 if and only if it has the WLP in degree 1 in every odd characteristic.

Moreover, if $\dim A_1 \leq \dim A_2$, then A has the WLP in characteristic zero if and only if every connected component of G contains either a loop or an odd cycle.

A polynomial ring for both perspectives

Let $I \subset R = k[x_1, \ldots, x_n]$ be a monomial ideal such that $A = R/I$ is artinian. We set

$$R_I = \mathbb{C}[t_m | m \text{ a monomial of } R,\ m \notin I]$$

Both objects mentioned before sit inside R_I when we take M to be the multiplication maps $\times L : A_i \to A_{i+1}$.

In particular, we can describe the WLP of A in terms of the analytic spread of ideals of R_I.

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Example

If $I = (x_1^2, \ldots, x_n^2)$, then

$$R_I = \mathbb{C}[t_m | m \text{ squarefree monomial in } n \text{ variables}]$$

Mixed Multiplicities of ideals

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$$R(\mathfrak{m}|J) = \bigoplus \mathfrak{m}^u J^v / \mathfrak{m}^{u+1} J^v$$

is a polynomial of the form (for $u, v \gg 0$)

$$\sum_{i=0}^{n} e_{(n-i,i)}(\mathfrak{m}|J) u^{n-i} v^i + \text{ terms of lower degree}$$

The nonnegative numbers $e_{(n-i,i)}(\mathfrak{m}|J)$ are called the mixed multiplicities of $\mathfrak{m}$ and J.

WLP and Mixed Multiplicities: Columns as sums

Theorem (Trung, 2001)

$$e_{(n-i,i)}(\mathfrak{m}|J) > 0 \iff 0 \leq i \leq \ell(J) - 1$$

where $\ell(J)$ is the analytic spread of J

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Theorem (Huh, 2012)

Let h be a product of linear forms in $\mathbb{C}[x_0, \ldots, x_n]$ and let J_h be the jacobian ideal of h. Then the coefficients of the characteristic polynomial of the hyperplane arrangement defined by h are the mixed multiplicities of J_h (after a convolution)

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Combining both results we conclude (over a field of characteristic zero):

Corollary

The analytic spread of the jacobian ideal of a product of linear forms h is equal to the rank of the matrix where the entries in each column are the coefficients of each linear form that divides h

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$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \leftrightarrow h = (a+d)(b)(c+d) \in \mathbb{C}[a,b,c,d]$$

means rank $M = 3 = \ell(J_h)$, where
$J_h = \left(b\,c + b\,d,\ a\,c + a\,d + c\,d + d^2,\ a\,b + b\,d,\ a\,b + b\,c + 2\,b\,d\right)$

Divisibility matrix and linear forms

Let $A = R/I$ be artinian, where $R = k[x_1, \ldots, x_n]$ (k has characteristic 0) and I is a monomial ideal.

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For every nonzero monomial m of degree i of A, define:

$$l_m = \sum_{\deg m' = i+1, m | m'} t_{m'} \in R_I$$

and

$$h_i = \prod_{\deg m = i} l_m \in R_I$$

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Theorem

A has the WLP in degree i if and only if $\ell(J_{h_i}) = \min(\dim A_i, \dim A_{i+1})$

WLP and Mixed Multiplicities: Rows as products

Let M be a $r \times s$ matrix with integer entries and constant row sum. The following is a well known result from Erhart theory:

Theorem

Let $\alpha_1, \ldots, \alpha_r$ be the rows of M. Then the analytic spread of $(x^{\alpha_1}, \ldots, x^{\alpha_r}) \subset k[x_1, \ldots, x_s]$ is the rank of M.

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$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \leftrightarrow (x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4) \subset k[x_1, x_2, x_3, x_4]$$

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The multiplication maps $\times L : A_i \to A_{i+1}$ *have constant row sum for every* i*, the row sum is always equal to* $i + 1$*.*

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Proposition

The h-vector of A is the f-vector of Δ

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The WLP of these algebras has been studied before:

1. (WLP in char 2) In terms of simplicial cohomology (Migliore, Nagel and Schenck)
2. (WLP in degree 1 and char 0) In terms of signless incidence matrices of graphs (Dao, Nair)

A small detour: incidence matrices everywhere

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- Dao and Nair studied the WLP in degree 1 when the field has characteristic 0. One of the steps in their characterization is noticing that the first multiplication map is the signless incidence matrix of the 1-skeleton of Δ, and the following well known result from graph theory:

Theorem

The rank of the signless incidence matrix of a graph G of n vertices is $n - b_G$, where b_G is the number of bipartite connected components of G.

Incidence matrices everywhere: j-multiplicity of clutters

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Theorem (Alilooee, Soprunov, Validashti (2017))

The j-multiplicity of the edge ideal of a graph G is positive if and only if every connected component of G contains an odd cycle.

The proof of this (very) particular case uses the theorem mentioned from graph theory

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Possible consequences for SLP

In the more general cases, they need combinatorial descriptions of ranks of matrices of the form $\times L^d : A_1 \to A_{d+1}$

Incidence matrices everywhere: Birational combinatorics

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Given a set of monomials $F = \{x^{\alpha_1}, \ldots, x^{\alpha_s}\}$ we call the matrix M with rows $\alpha_1, \ldots, \alpha_s$ the log-matrix of F.

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Theorem ((DPB), Simis and Villarreal (2003))

Let X_d be the set of all monomials of degree d in $k[x_1, \ldots, x_n]$, and let $F \subset X_d$.

Then the extension $k[F] \subset k[X_d]$ is birational if and only if the gcd of all maximal minors of the log-matrix of F is d

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WLP in positive characteristics

The prime divisors of the gcd of all the maximal minors are exactly the characteristics where WLP fails.

Incidence matrices everywhere: Birational combinatorics

Other statements include:

Theorem ((DPB), Simis and Villarreal)

A rational map $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ defined by monomials of degree d is birational if and only if the determinant of the log-matrix of the monomials defining the map is $\pm d$.

Theorem (Simis and Villarreal (2003))

If the log-matrix of a set of monomials F has full rank and the ideal (F) has a linear presentation, then the extension $k[F] \subset k[X_d]$ is birational.

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Positive characteristics

In particular, WLP in characteristic 0 + a specific ideal having linear presentation implies WLP in positive characteristics

Incidence matrices everywhere: Birational combinatorics

Note that given a $r \times s$ matrix M with integer entries and constant row sum d and $r \geq s$, we can add every column to the last one, so the maximal minors are always divisible by d.

$$\det \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \det \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 2 \end{pmatrix} = 2 \det \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Incidence matrices everywhere: Birational combinatorics

Given a rational monomial map $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ defined by monomials of degree 2, we can associate a graph, by taking the coordinates to be edges (possibly loops), the vertices being the variables.

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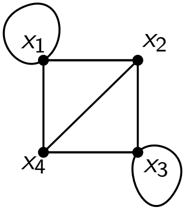

	x_1	x_2	x_3	x_4
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Incidence matrices everywhere: Birational combinatorics

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- $\det M \neq 0$, *where M is the log-matrix of the set of monomials defining φ_G*

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Question

What is the connection between ideals of linear type and the WLP in positive characteristics?

Consequences for the WLP of monomial ideals (in degree 1)

The only difference between the log-matrix of a set of monomials of degree 2 and the incidence matrix of a graph is that for incidence matrices, loops are rows with only one nonzero entry, which is 1, while for log-matrices that entry is 2. We can then show the following:

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An example

Let $R = k[x_1, x_2, x_3, x_4]$ and $I = (x_1^3, x_2^2, x_3^3, x_4^2, x_1 x_3)$.

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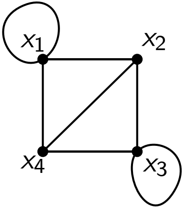

Then $A = R/I$ has the WLP in degree 1 in every characteristic that is not 2 by the theorem.

Mixed multiplicities and the WLP in higher degrees

To generalize these ideas to higher degrees we use the results from Trung and Verma (2007) connecting mixed volumes and mixed multiplicities.

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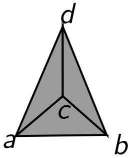
We want to connect the positivity of the last mixed multiplicity of an ideal to the WLP in characteristic 0, and the value of this mixed multiplicity to be a bound on the characteristics where the WLP can fail.

Incidence ideals: example

Consider the following simplicial complex:



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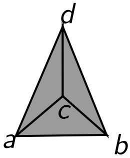

Then $A = k[a,b,c,d]/(a^2, b^2, c^2, d^2, abd)$. The multiplication matrix that determines the WLP in degree 1 and the corresponding ideal are:

$$A_1 \xrightarrow{\begin{array}{c} \\ ab \\ ac \\ ad \\ bc \\ bd \\ cd \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}\end{array}} A_2, \quad I_\Delta(1) = (\underbrace{t_a t_b}_{ab}, \underbrace{t_a t_c}_{ac}, \underbrace{t_a t_d}_{ad}, \underbrace{t_b t_c}_{bc}, \underbrace{t_b t_d}_{bd}, \underbrace{t_c t_d}_{cd})$$

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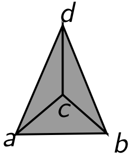

Then $A = k[a, b, c, d]/(a^2, b^2, c^2, d^2, abd)$. The multiplication matrix that determines the WLP in degree 2 and the corresponding ideal are:

$$A_2 \xrightarrow{\begin{array}{c|cccccc} & ab & ac & ad & bc & bd & cd \\ abc & 1 & 1 & 0 & 1 & 0 & 0 \\ acd & 0 & 1 & 1 & 0 & 0 & 1 \\ bcd & 0 & 0 & 0 & 1 & 1 & 1 \end{array}} A_3$$

$$I_\Delta(2) = (\underbrace{t_{ab}t_{ac}t_{bc}}_{abc}, \underbrace{t_{ac}t_{ad}t_{cd}}_{acd}, \underbrace{t_{bc}t_{bd}t_{cd}}_{bcd})$$

A simple example of linear type

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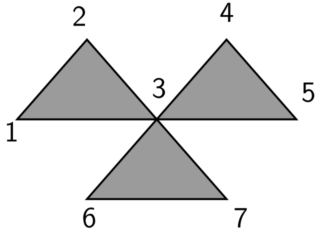

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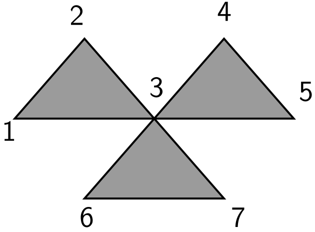
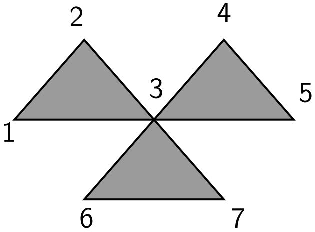


By the results of Faridi and Alilooee the facet ideal of the complex above is of linear type.

A simple example of linear type



Taking each vertex to be an edge of a new simplicial complex Δ, we see that the log-matrix of the set of generators of the facet ideal of the complex above: $(x_1x_2x_3, x_3x_4x_5, x_3x_6x_7)$ is the multiplication map

$$\times L : A_2 \to A_3$$

where $A = k[a, b, c, d, e]/((a^2, b^2, c^2, d^2, e^2) + (ad, ae, de))$.

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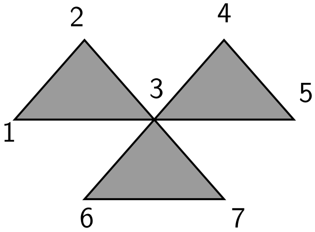

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In particular, we can check that A has the WLP in degree 2 in every characteristic.