Three point functions, balanced manifolds and incidence toric ideals

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Joint work with Barbara Betti, Sean Grate and Flavio Salizzoni

A physics motivation

1. n massless particles $p_1, \ldots, p_n$ in $\mathbb{R}^d$
2. $p_{ij} = p_i \cdot p_j$ (Lorentzian inner product).
3. $P = (p_{ij})$ is a symmetric $n \times n$ matrix with zero diagonal.
4. Simplified form of 3-point functions: $C_{ijk} = \frac{1}{p_{ij} p_{ik} p_{jk}}$ (topology?)
5. Relations between C_{ijk} are the same as relations between $c_{ijk} = p_{ij} p_{ik} p_{jk}$ (toric ideals!)

$$\varphi_{n,3,2} : \mathbb{C}[c_{ijk} : 1 \leq i < j < k \leq n] \longrightarrow \mathbb{C}[p_{ij} : 1 \leq i < j \leq n]$$

$$\varphi_{n,3,2}(c_{ijk}) = p_{ij} p_{ik} p_{jk}$$

Incidence toric ideals

Incidence toric ideals

Let $n > k > t$ be integers. The incidence toric ideal $I_{n,k,t}$ is the kernel of the map

$$\varphi_{n,k,t} : \mathbb{C}[c_A : A \in \binom{[n]}{k}] \longrightarrow \mathbb{C}[p_B : B \in \binom{[n]}{t}]$$

$$\varphi_{n,k,t}(c_A) = \prod_{B \in \binom{A}{t}} p_B$$

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$I_{n,2,1}$ is the toric edge ideal of the complete graph K_n

What is a bipartite hypergraph?

Theorem ((Villarreal 95, Ohsugi-Hibi 99))

The ideal $I_{n,2,1}$ is generated by binomials given by primitive even closed walks in K_n.

Theorem (Petrović and Stasi, 2014)

If H is a hypergraph, a Markov basis of the toric ideal I_H is given by monomial walks in H.

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Toric ideals of graphs map edges to vertices. This is analogous to $I_{n,k,k-1}$

Question

What is right notion of "even" for incidence toric ideals?

Answer?: balancedness in topology!

Boundary maps without signs

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

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$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

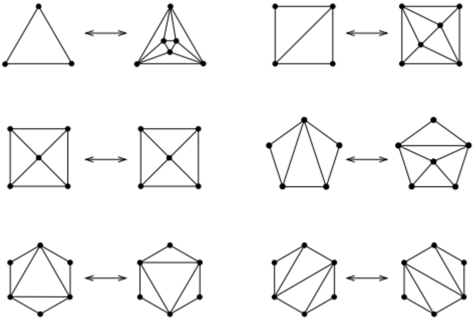
Theorem (Betti-Grate-H-Salizzoni, 2026+)

Let Δ be a triangulation of an orientable d-manifold such that the graph of Δ is $(d+1)$–colorable. Then the orientation of Δ gives a primitive binomial in $I_{n,d+1,d}$.

Balanced bistellar flips: how to find elements in $I_{n,k,t}$

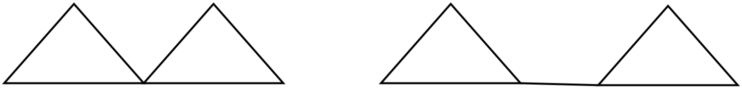
Question

If we know all the elements in $I_{n_0,k,t}$, how can we generate new elements in $I_{n,k,t}$ for $n > n_0$?



(Izmestiev, Klee, Novik 2018)

An intuition for elements in $I_{n,k,t}$ that are not manifolds



We can glue two nonbalanced manifolds and get a binomial in $I_{n,k,t}$, just as we can glue two odd cycles and get an element in $I_{n,2,1}$

The dimension: same matrix, different interpretations

1. Design theory (null-designs)
2. Algebraic geometry (Hard Lefschetz theorem for $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$)
3. Symmetric polynomials (Specht polynomials of shape $(n-k, k)$)
4. and more

Theorem (Betti-Grate-H-Salizzoni, 2026+)

For every $n > k > t$, then

$$\dim \frac{R}{I_{n,k,t}} = \min\left(\binom{n}{k}, \binom{n}{t}\right)$$

Proof idea: the corresponding matrix always has full rank

A polytope and some of its combinatorics

Let $\mathcal{P}_{n,k,t}$ be the convex hull of the columns of the incidence matrix of k and t-subsets of $[n]$.

$$
\begin{array}{c} \\ 12 \\ 13 \\ 23 \\ 14 \\ 24 \\ 34 \end{array}
\begin{array}{c} \begin{array}{cccc} 123 & 124 & 134 & 234 \end{array} \\
\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}
$$

$n = 4, k = 3, t = 2$

Theorem (Betti-Grate-H-Salizzoni, 2026+)

The polytope $\mathcal{P}_{n,k,t}$ is $(2^t - 1)$–neighborly. In other words, every set of $2^t - 1$ vertices of $\mathcal{P}_{n,k,t}$ forms a simplicial face of $\mathcal{P}_{n,k,t}$.

Proof idea: Identify kernel elements with null-designs

A polytope and some of its geometry

Theorem (Betti-Grate-H-Salizzoni, 2026+)

Let f be the function such that $f(n,k,t) = Vol(\mathcal{P}_{n,k,t})$, where Vol is the normalized volume with respect to the euclidean lattice. Then f is not a polynomial function.

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$Vol(\mathcal{P}_{6,3,2}) = 162$

$Vol(\mathcal{P}_{7,3,2}) = 85368$

but our methods only give $6 \leq Vol(\mathcal{P}_{7,3,2})$

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Theorem (Betti-Grate-H-Salizzoni, 2026+)

Let $J_{n,k}$ be the ideal generated by the binomials arising from every $(k-1)$–dimensional crosspolytope on vertex set $A \subseteq [n]$. Then

$$J_{n,k} = I_{n,k,k-1} : m^{\infty} = \{f \ : \ fm^b \in I_{n,k,k-1} \quad \textit{for some } b\},$$

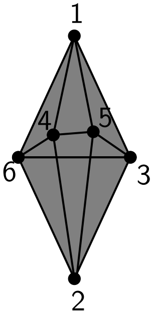
where m is the product of all variables $c_{ij\ell}$.

Proof idea: orientations of crosspolytopes are known to generate kernel of incidence matrix between k and $k-1$ subsets of $[n]$

A combinatorial way of viewing crosspolytopes and $J_{2k,k}$: Specht polynomials of (k, k)

1	3	5
2	4	6

$\updownarrow$



$$(x_1 - x_2)(x_3 - x_4)(x_5 - x_6)$$

$$x_1x_3x_5 + x_2x_4x_5 + x_2x_3x_6 + x_1x_4x_6 - x_2x_3x_5 - x_1x_4x_5 - x_1x_3x_6 - x_2x_4x_6$$

$$c_{135}c_{245}c_{236}c_{146} - c_{235}c_{145}c_{136}c_{246} \in J_{6,3}$$