Lefschetz properties and analytic spread

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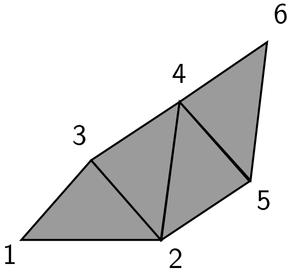
Stanley-Reisner, Facet (and incidence) ideals

A simplicial complex Δ on vertex set $[n]$ is a collection of subsets Δ of $[n]$ such that $\tau \subset \sigma \in \Delta \implies \tau \in \Delta$. We write $\Delta = \langle F_1, \ldots, F_s \rangle$ if $F_1, \ldots, F_s$ are the facets (maximal subsets) of Δ.



$$\Delta = \langle \{1, 2, 3\}, \{2, 3, 4\}, \{2, 4, 5\}, \{5, 4, 6\} \rangle$$

Stanley-Reisner, Facet (and incidence) ideals

Let $S = k[x_1, \ldots, x_n]$ and $\Delta = \langle F_1, \ldots, F_s \rangle$ a simplicial complex with vertex set $[n]$.

- The **Stanley-Reisner** ideal of Δ is the ideal

$$\mathcal{N}(\Delta) = (\prod_{i \in B} x_i : B \notin \Delta) \subset S$$

- The **Facet** ideal of Δ is the ideal

$$\mathcal{F}(\Delta) = (\prod_{i \in F_1} x_i, \ldots, \prod_{i \in F_s} x_i) \subset S$$

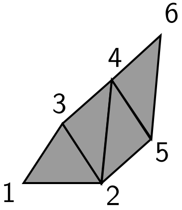
Both constructions give bijections between simplicial complexes and squarefree monomial ideals

Stanley-Reisner, Facet (and incidence) ideals

$$\mathcal{N}(\Delta) = (\prod_{i \in B} x_i : B \notin \Delta), \quad \mathcal{F}(\Delta) = (\prod_{i \in F_1} x_i, \dots, \prod_{i \in F_s} x_i)$$

$(x_1x_4, x_1x_5, x_3x_5, x_1x_6, x_2x_6, x_3x_6)$ ⟷

$\mathcal{N}(\Delta)$



Δ

↕

$(x_1x_2x_3, x_2x_3x_4, x_2x_4x_5, x_4x_5x_6)$

$\mathcal{F}(\Delta)$

Lefschetz properties

Let I be a monomial ideal of $S = k[x_1, \ldots, x_n]$ such that $A = S/I$ is artinian, and $L = x_1 + \cdots + x_n \in S_1$.

Definition

We say A satisfies the **weak Lefschetz property (WLP)** if the multiplication maps

$$\times L : A_i \to A_{i+1}$$

have full rank for every i.
If moreover the maps

$$\times L^j : A_i \to A_{i+j}$$

have full rank for every i, j, we say A satisfies the **strong Lefschetz property (SLP)**

A motivation from Combinatorics

Proposition

If A is an algebra that satisfies the WLP, then

$$\dim A_1 \leq \dim A_2 \leq \cdots \leq \dim A_k \geq \cdots \geq \dim A_d$$

for some k, in other words, the h-vector of A is **unimodal**.

We are particularly interested in algebras of the form:

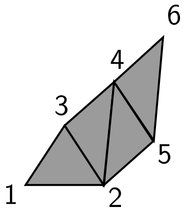
$$A(\Delta) = \frac{S}{(\mathcal{N}(\Delta), x_1^2, \ldots, x_n^2)}$$

where Δ is a simplicial complex.

$$\dim A(\Delta)_i = f_{i-1} = \text{the number of } i-1 \text{ dimensional faces of } \Delta$$

An example with the SLP

$(x_1x_4, x_1x_5, x_3x_5, x_1x_6, x_2x_6, x_3x_6) \longleftrightarrow$



$\mathcal{N}(\Delta)$

Δ

The algebra

$$A(\Delta) = k[x_1, \ldots, x_6]/(\mathcal{N}(\Delta), x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2)$$

has the SLP whenever k is not a field of characteristic 2.

Analytic spread of monomial ideals (via toric ideals)

Given a monomial ideal $I = (m_1, \ldots, m_s) \subset S = k[x_1, \ldots, x_n]$ where $\deg m_i = d$ for all i, define

$$\begin{aligned} \varphi\colon k[e_1, \ldots, e_s] &\to S \\ e_i &\mapsto m_i. \end{aligned}$$

The **analytic spread of** I is

$$\ell(I) = \dim \frac{k[e_1, \ldots, e_s]}{\ker \varphi}$$

Analytic spread and rank

Using the theory of Rees algebras it is possible to define the analytic spread of an arbitrary ideal. Even in the general case, analytic spread is known to be related to ranks of special matrices

- (Simis, 2003): Analytic spread as rank of jacobian matrices

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- Analytic spread as ranks of matroids

Analytic spread and positivity of algebraic invariants

A classical fact from Commutative algebra

Let $I = (f_1, \ldots, f_s) \subset k[x_1, \ldots, x_n]$, then

$$\ell(I) \leq \min\{s, n\}$$

When $\ell(I) = n$, the ideal I is said to have **maximal analytic spread**

Analytic spread and positivity of algebraic invariants

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The motivation for maximal analytic spread

Many theorems in Commutative Algebra relate positivity of algebraic invariants to ideals of maximal analytic spread

Analytic spread and positivity of algebraic invariants

- (Trung, 2006) All the mixed multiplicities of an ideal are positive if and only if it has maximal analytic spread
- (Nishida and Ulrich, 2010) j-multiplicity: $j(I) > 0 \iff \ell(I) = n$
- (Cutkosky, 2011) ε-multiplicity: $\varepsilon(I) > 0 \iff \ell(I) = n$
- (Morey and Villarreal, 2012) Symbolic defect: If I is a squarefree monomial ideal, then $\ell(I) = n \implies \text{sdefect}(I, m) > 0$ for some m
- (Castillo, Cid-Ruiz, Li, Montaño and Zhang, 2020) All the mixed multiplicities of a sequence of ideals are positive if and only if the ideals have maximal analytic spread

The main idea

$$\begin{aligned} \times L^j : A_i \to A_{i+j} \text{ has full rank} &\implies \text{a matrix has full rank} \\ &\implies \text{an ideal has maximal analytic spread} \\ &\implies \text{a number is positive} \end{aligned}$$

In particular, if an artinian algebra A has the WLP, a sequence of numbers has no (internal) zeros

Analytic spread of skeleta

Theorem (-, 2024)

Let Δ be a pure simplicial complex on n vertices of dimension d, and Δ_k be its k-skeleton for $0 \leq k < d$. Then

$$\ell(\mathcal{F}(\Delta_k)) = n$$

Idea of proof: Hard Lefschetz theorem for product of projective spaces

Eulerian numbers: a simplicial point of view

Let

$$\Delta_{n,k} = \text{conv}\Big(\sum_{i \in I} e_i : I \subset 2^n \;\; |I| = k \Big)$$

Eulerian numbers

The numbers $A(n,k) = \text{vol}(\Delta_{n,k})$ are called **Eulerian numbers**

A fact about Eulerian numbers

For every n, the sequence

$$A(n,1), \ldots, A(n,n)$$

is real rooted and has no zeros

Simplicial Eulerian numbers

From the simplex to simplicial complexes

Let Δ be a d-dimensional pure simplicial complex. For every $0 \leq k < d$ define the polytope

$$P_{\Delta,k} = \mathrm{conv}(e_{i_1} + \cdots + e_{i_{k+1}} : \{i_1, \ldots, i_{k+1}\} \in \Delta)$$

Note that when Δ is a simplex, $P_{\Delta,k}$ is a hypersimplex for every k

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For every d-dimensional pure simplicial complex Δ on n vertices ,

$$A_\Delta(k) = vol(P_{\Delta,k}) > 0 \quad 0 \leq k < d$$

where vol is the normalized volume

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- (when) Is the sequence $A_\Delta(0), \ldots, A_\Delta(d-1)$ unimodal/log-concave/real rooted?

Symbolic powers

Symbolic powers of squarefree monomial ideals

Let $\mathcal{F}(\Delta) \subset S = k[x_1, \ldots, x_n]$ be a squarefree monomial ideal where $\Delta = \langle F_1, \ldots, F_s \rangle$. The m-th symbolic power of $\mathcal{F}(\Delta)$ is:

$$\mathcal{F}(\Delta)^{(m)} = \bigcap_{P \in \mathcal{P}} P^m$$

where

$$\mathcal{P} = \{(x_i \colon i \notin F_1), \ldots, (x_i \colon i \notin F_s)\}$$

If $\mathcal{F}(\Delta) = (x_1x_2, x_2x_3, x_1x_3)$, then

$$\mathcal{F}(\Delta)^{(2)} = (x_1x_2x_3, x_1^2x_2^2, x_2^2x_3^2, x_1^2x_3^2) \neq \mathcal{F}(\Delta)^2$$

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The intuition for symbolic powers of facet ideals

Symbolic powers can capture odd cycles in Δ (as a hypergraph)

The symbolic defect: a horizontal perspective

Symbolic Defect sequence of an ideal (GGSVT, 2018)

Let I be an ideal, define

$$\text{sdefect}(I, m) = \text{ the minimal number of generators of } I^{(m)}/I^m$$

for every m.

Theorem (GGSVT, 2018)

Let Δ be the simplex on n vertices and Δ_k the k-skeleton. Then

$$\text{sdefect}(\mathcal{F}(\Delta_d), 2) = \binom{n}{d+1}$$

In other words, $\text{sdefect}(\mathcal{F}(\Delta_d), 2)$ *is the number of d-faces of the simplex on n vertices.*

Symbolic defect polynomials

$$\text{sdefect}(\mathcal{F}(\Delta), m)$$

$$\vdots$$

$$\text{sdefect}(\mathcal{F}(\Delta), 4)$$

$$\text{sdefect}(\mathcal{F}(\Delta), 3)$$

$$\text{sdefect}(\mathcal{F}(\Delta), 2)$$

Symbolic defect polynomials

$$\begin{array}{cccc}
\mathrm{sdefect}(\mathcal{F}(\Delta_1), m) & \cdots & \mathrm{sdefect}(\mathcal{F}(\Delta_{d-1}), m) & \mathrm{sdefect}(\mathcal{F}(\Delta), m) \\
\vdots & \ddots & \vdots & \vdots \\
\mathrm{sdefect}(\mathcal{F}(\Delta_1), 4) & \cdots & \mathrm{sdefect}(\mathcal{F}(\Delta_{d-1}), 4) & \mathrm{sdefect}(\mathcal{F}(\Delta), 4) \\
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\end{array}$$

Symbolic defect polynomials

The second symbolic defect polynomial

The **second symbolic defect polynomial** of a pure simplicial complex Δ is:

$$\mu(\Delta, 2, x) = \sum_{i \geq 1} \text{sdefect}(\mathcal{F}(\Delta_i), 2) x^{i+2}$$

Symbolic defect polynomials

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Theorem (-, 2024)

Let Δ be a flag simplicial complex.

- *The coefficient of x^3 in $\mu(\Delta, 2, x)$ is equal to the number of triangles of Δ.*
- *The sequence of coefficients of $\mu(\Delta, 2, x)$ is bounded below by the f-vector of Δ.*

An example

Let $\mathcal{N}(\Delta) = (x_i x_{i+1} : 1 \leq i \leq 14) \subset k[x_1, \ldots, x_{15}]$. Then

$$\mu(\Delta, 2, x) = 286x^3 + 495x^4 + 462x^5 + 210x^6 + 36x^7 + x^8$$

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and the f-vector of Δ is:

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- When Δ is the independence complex of a forest, do we always have equality? (true up to 16 vertices!)
- When is the second symbolic defect polynomial unimodal?