Powers of a simplex: Resolutions meet Partititions

Thiago Holleben

Dalhousie University

Joint work with Trung Chau, Art Duval, Sara Faridi, Susan Morey and Liana Şega

January 27, 2024

The starting point: Resolutions

Given an ideal $I = (m_1, \ldots, m_s)$ in a polynomial ring R, there is a map

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By repeating this process, we get a chain complex

$$0 \to R^{\beta_p} \to R^{\beta_{p-1}} \to \cdots \to R^{\beta_1} \to I \to 0$$

where $\beta_1 = s$

This chain complex is called a **resolution** of I. The numbers β_i are called **betti numbers** of I

From Algebra to Topology: (De)homogenizing complexes

Let $I = (x, y, z) \subset \Bbbk[x, y, z]$. The following complex is a (minimal) resolution of I:

$$0 \to R \xrightarrow{\begin{pmatrix} z \\ -y \\ x \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} -y & -z & 0 \\ x & 0 & -z \\ 0 & x & y \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} x & y & z \end{pmatrix}} I \to 0$$

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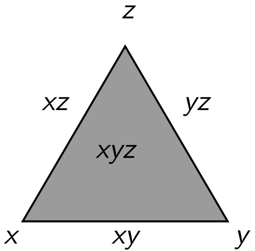
Replacing every variable with 1 we get:

$$0 \to \Bbbk \xrightarrow{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}} \Bbbk^3 \xrightarrow{\begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}} \Bbbk^3 \xrightarrow{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}} \Bbbk \to 0$$

which is the chain complex of the 2-simplex (a triangle)

From Topology to Algebra: Taylor's resolution

Let $I = (x, y, z)$ be the same ideal and Δ the 2-simplex. We can label the vertices of Δ by the generators of I, edges by the lcm's of the labels of the vertices and so on.



If we take the chain complex of the simplex above (with labelings) we get the (Taylor) resolution of I

Powers of a simplex

The $r = 1$ case

I a monomial ideal $\implies$ Labeled simplex gives a (sometimes minimal) resolution of I

Upshot: Number of faces of a simplex gives a sharp upper bound on betti numbers of monomial ideals

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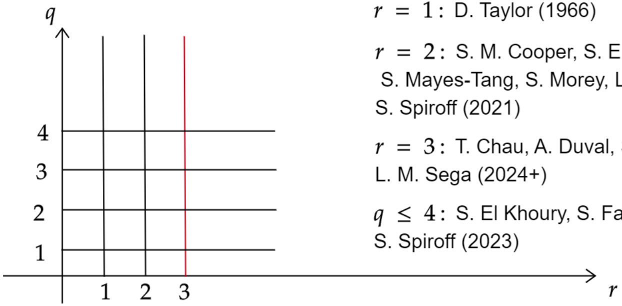
If we replace I by I^r on the algebra side, what replaces the simplex? Is this object simplicial? Polyhedral?

The main question

Is there a complex Δ_q^r such that it supports the resolution of the r-th power of any monomial ideal I generated on q elements, while also giving the minimal resolution for some I^r?

What is known: $r = 1, 2$, $q \leq 4$



$r = 1$: D. Taylor (1966)

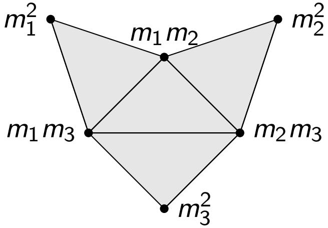
$r = 2$: S. M. Cooper, S. El Khoury, S. Faridi, S. Mayes-Tang, S. Morey, L. M. Şega and S. Spiroff (2021)

$r = 3$: T. Chau, A. Duval, S. Faridi, -, S. Morey, L. M. Sega (2024+)

$q \leq 4$: S. El Khoury, S. Faridi, L. M. Şega and S. Spiroff (2023)

A picture of $r = 2$, $q = 3$

$$I = (m_1, m_2, m_3)$$



The minimal nonfaces for $r = 3$

$$V(\Delta_q^3) = \{v \in \mathbb{N}^q \colon v_1 + \cdots + v_q = 3\}$$

Theorem (CDF-MS, 2024+)

The minimal nonfaces of Δ_q^3 for $q \geq 5$ are (up to permutation)

1. $(3)0^{q-1}, (0,3)0^{q-2}$
2. $(3)0^{q-1}, (0,2,1)0^{q-3}$
3. $(3)0^{q-1}, (0,1,1,1)0^{q-4}$
4. $(3)0^{q-1}, (1,2)0^{q-2}$
5. $(3)0^{q-1}, (1,1,1)0^{q-3}$
6. $(2,1)0^{q-2}, (0,1,2)0^{q-3}$
7. $(2,1)0^{q-2}, (0,1,1,1)0^{q-4}$
8. $(2,1)0^{q-2}, (0,0,2,1)0^{q-4}$
9. $(2,1)0^{q-2}, (1,0,2)0^{q-3}$
10. $(2,1)0^{q-2}, (1,2)0^{q-2}, (0,0,1,1,1)0^{q-5}$

The vertices and some symmetries

The vertex set of the (simplicial) complex Δ_q^3 we are looking for is the following:

$$I = (m_1, m_2, m_3), \text{the generator } m_1^2 m_2 \text{ of } I^3 \leftrightarrow (2,1,0) \in \Delta_3^3$$

Note that reordering the generating set of I does not affect Δ_q^3, and in particular

$$\{v_1, \dots, v_s\} \in \Delta_q^3 \iff \{\sigma v_1, \dots, \sigma v_s\} \in \Delta_q^3$$

for any permutation $\sigma \in S_q$

From a graph of monomials to a graph of partitions

Theorem (EK,F,S,S (2023))

After dehomogenizing, the first step of the minimal resolution of the r-th power of any monomial ideal is contained in the first step of the chain complex of a simplicial complex $\mathbb{S}_q^r$. This simplicial complex has vertex set

$$V(\mathbb{S}_q^r) = \{v \in \mathbb{N}^q \colon v_1 + \cdots + v_q = r\}$$

Theorem (CDF-MS, 2024+)

$$\Delta_q^3 = \mathbb{S}_q^3$$

An example: the edge $(3,0),(2,1)$ in $\mathbb{S}_2^3$

$a = (3,0)$ and $b = (2,1)$

$$\sum_{i \in A} x_i \leq \max(\sum_{i \in A} a_i, \sum_{i \in A} b_i), \qquad x_1 + x_2 = 3$$

$$x_1 \leq 3 \;\; (A = \{1\}), \quad x_2 \leq 1 \;\; (A = \{2\}), \quad x_1 + x_2 = 3$$

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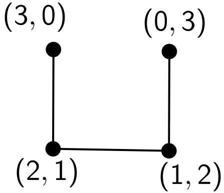
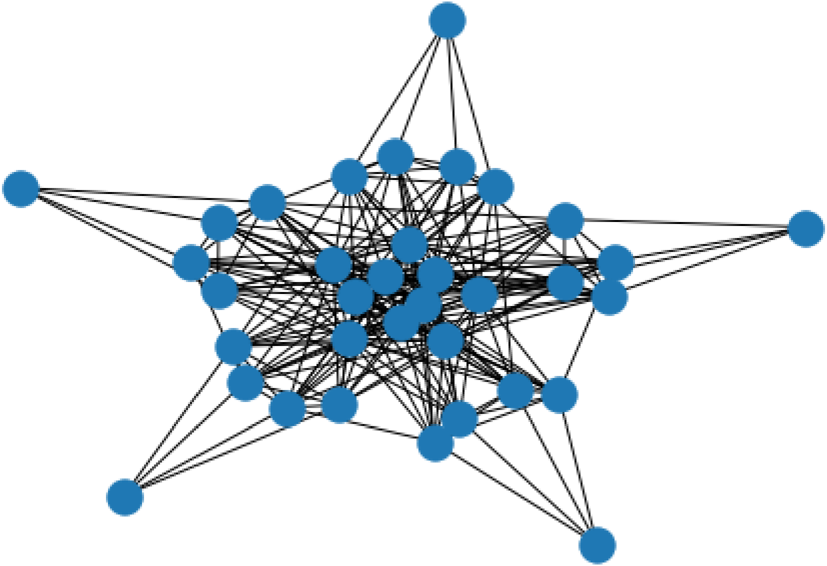
$(3,0,0)$ and $(0,2,1)$ do not form an edge, $(1,1,1)$ is a solution to the system of equations $x_1 + x_2 + x_3 = 3$,

$$x_1 \leq 3, \quad x_2 \leq 2, \quad x_3 \leq 1, \quad x_1 + x_2 \leq 3, \quad x_1 + x_3 \leq 3, \quad x_2 + x_3 \leq 3$$

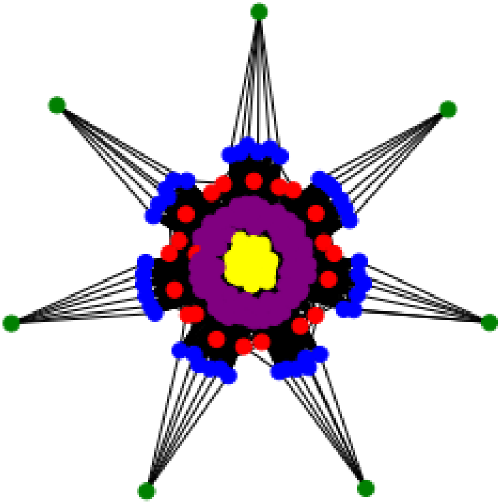
Visualizing $r = 3$, $q = 2$



The 1-skeleton of $\Delta_5^3 = \mathbb{S}_5^3$



The graph for $r = 4, q = 7$



A graph of partitions

Given two partitions $\lambda = \lambda_1 \dots \lambda_s$ and $\mu = \mu_1 \dots \mu_t$ of r, consider the vectors

$$\lambda' = (\lambda_1, \dots, \lambda_s, \underbrace{0, \dots, 0}_{t \text{ times}}), \quad \mu' = (\underbrace{0, \dots, 0}_{s \text{ times}}, \mu_1, \dots, \mu_t)$$

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$$1 \sim 1, \quad 11 \sim 11, \quad 21 \sim 111, \quad 211 \sim 211, \quad 31 \sim 1111$$

A graph of partitions

Let G_r be the graph with vertex set $V(G_r) = \{\text{partitions of } r\}$ and $\{\lambda, \mu\} \in E(G_r)$ if and only if $\lambda \sim \mu$.

G_r is a simple undirected graph with loops

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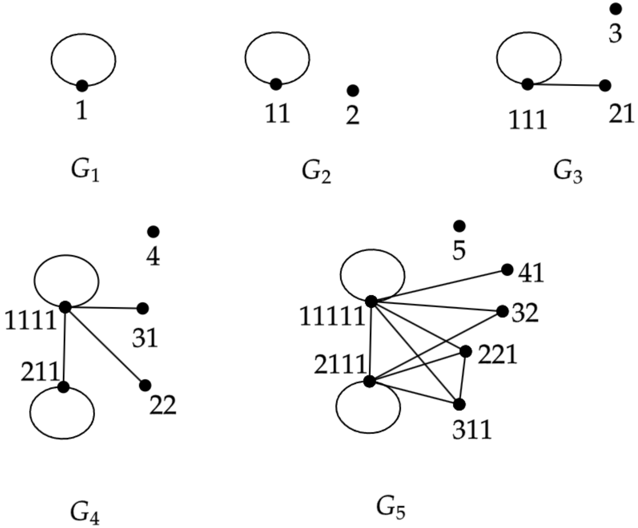
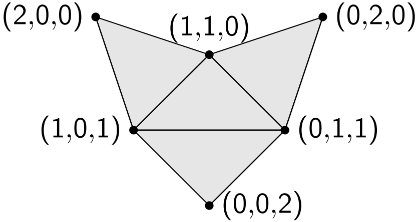
Upshot: If we know G_i for every $i \leq r$, we can give a sharp upper bound to $\beta_2(I^r)$, where I is a monomial ideal generated by q elements.

A graph of partitions: G_1, G_2, G_3, G_4, G_5



Visualizing partitions in the case $r = 2$, $q = 3$



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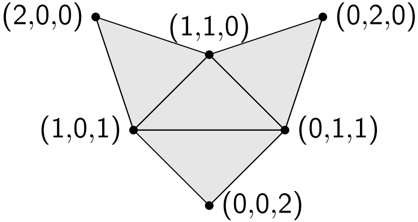

$$\{2, 11\} \notin E(G_2), \quad \{1, 1\} \in E(G_1)$$