Homological invariants of ternary graphs

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April 5, 2023

Independence Complexes

Given a graph $G = (V, E)$, we define its edge ideal

$$I(G) := (x_i x_j | \{i, j\} \in E)$$

and given a simplicial complex Δ, we define its Stanley-Reisner ideal

$$I_\Delta := (x_{i_1} \dots x_{i_s} | \{i_1, \dots, i_s\} \notin \Delta)$$

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Theorem (Hochster's formula)

Let Δ be a simplicial complex. Then

$$b_{i,x_\tau}(I_\Delta) = \dim \tilde{H}_{|\tau|-i-2}(\Delta_\tau; k)$$

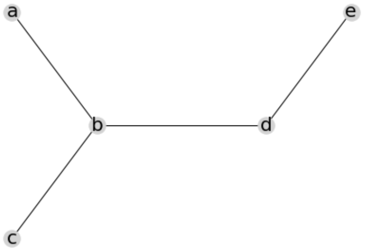
where Δ_τ is the restriction of Δ to the vertices in τ

Independence Complexes

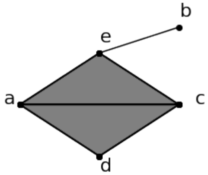
Let $R = k[x_1, \ldots, x_n]$, $I(G)$ the edge ideal of a graph G and I_Δ the Stanley-Reisner ideal of Δ.

Independence complex of G

A set $S \subset V(G)$ is a face of the simplicial complex $\text{Ind}(G)$ if and only if S is an independent set of G, that is, none of the edges of G are between elements of S.



(a) A graph G



(b) $\text{Ind}(G)$

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Useful facts

- If G has an isolated vertex, $\text{Ind}(G)$ is a cone.
- $I(G) = I_{\text{Ind}(G)}$

Ternary graphs

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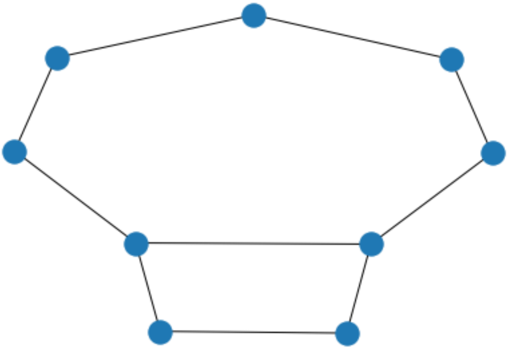

A ternary graph with a non-induced 9-cycle

Ternary graphs

Theorem (J. Kim, 2022)

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Corollary

The betti table of the edge ideal of a ternary graph does not depend on the characteristic of the base field.

Ternary graphs

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2. When $\text{Ind}(G)$ is not contractible, what is the dimension of the sphere $\text{Ind}(G)$ is homotopy equivalent to?

3. Can we describe projective dimension, depth and regularity of $S/I(G)$ in terms of G? (these invariants will be characteristic-free)

Setting some notation

Given a graph G and an independent subset $X \subset V(G)$, we set
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Let G be a graph, $X, Y \subset V(G)$ such that X is independent and $X \cap Y = \emptyset$. We denote by $G(X|Y)$ the graph $G - N[X] - Y$.

Forests

Theorem (M. Marietti and D. Testa, 2008)

Let G be a forest. Then $\mathrm{Ind}(G)$ is either contractible or homotopy equivalent to $S^{\gamma(G)-1}$, where $\gamma(G) = \min\{|S| \mid S \subset V(G),\ N[S] = V(G)\}$ is called the lower dominating number of G.

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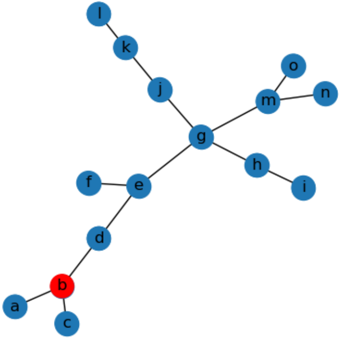
How can we determine if the independence complex of a forest is contractible?

A leaf-filtration

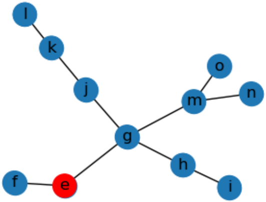
Consider the forest F below:



Note that b is adjacent to a leaf

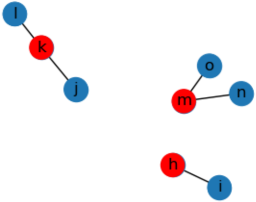
A leaf-filtration

Next note that e is adjacent to a leaf



Now note that k, m and h are adjacent to leaves

A leaf-filtration



After removing the vertices adjacent to k, m and h (and the 3 vertices) we get the empty graph.

A leaf-filtration

We call the sequence of subgraphs:

$$F(\emptyset|\emptyset), F(b|\emptyset), F(b, e|\emptyset), F(b, e, k|\emptyset), F(b, e, k, m|\emptyset), F(b, e, k, m, h|\emptyset) = \emptyset$$

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Theorem

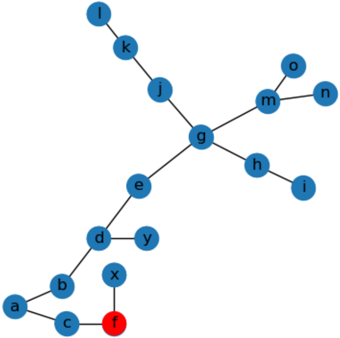
A forest F admits a leaf-filtration if and only if its independence complex is not contractible. Moreover, in that case the empty graph can be written as

$$F(X|\emptyset) = \emptyset$$

where X is the set of vertices adjacent to a leaf in each step. We then have

$$\mathrm{Ind}(F) \cong S^{|X|-1}$$

A forest that does not have a leaf-filtration



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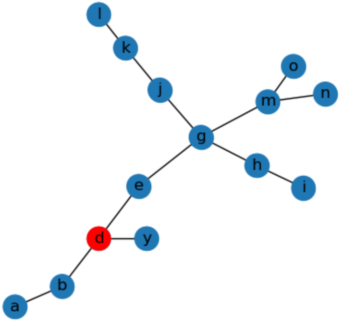

After this step, the vertex *a* is isolated

Back to ternary graphs

Let G be a ternary graph and $S \subset V(G)$ be such that $G(\emptyset|S) = G - S$ is a forest. Then whenever $A \subset S$ is an independent set

$$G(A|S\backslash A) = G - N[A] - S\backslash A$$

is also a forest.

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Definition

Let k be the number of forests of the form $G(A|S \backslash A)$ that have a non contractible independence complex. We call $i(G) = (-1)^k$ the sign of G.

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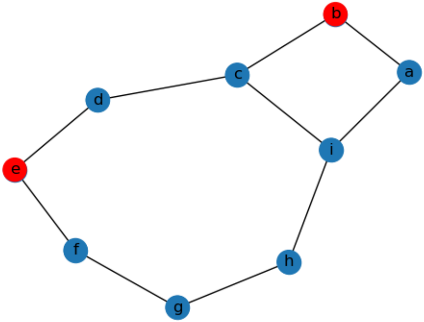
Theorem

The independence complex of a ternary graph G is contractible if and only if $i(G) = 1$.

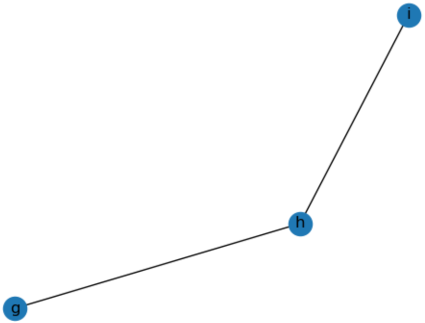
Example sign

Let G be the following graph and $S = \{e, b\}$



Example sign

The forests we get of the form $G(A|S \setminus A)$ are:



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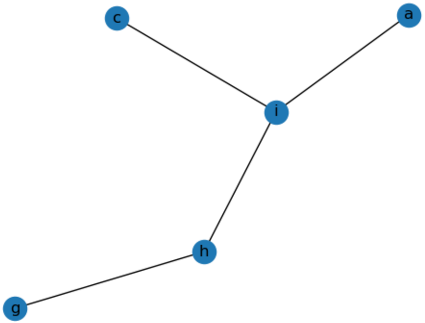

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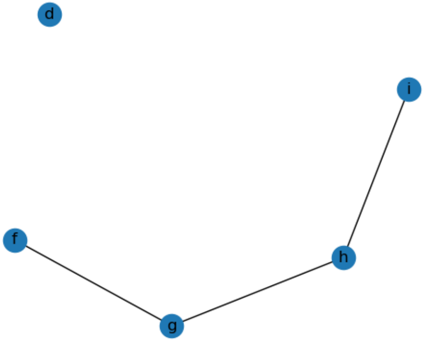

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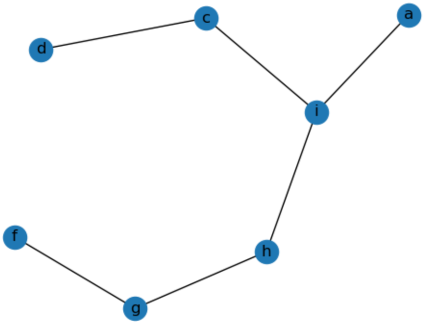
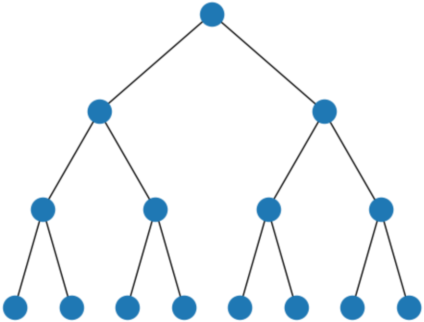


$G(\emptyset|b,e)$ has a contractible independence complex, so $i(G) = (-1)^1$

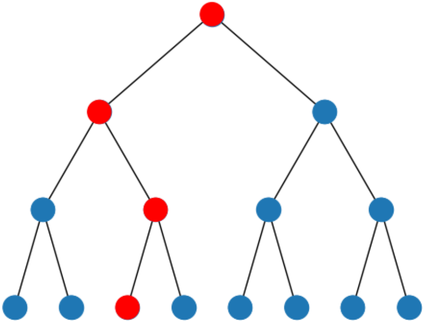
Filtrations

Let G be a ternary graph and $S = \{v_1, \ldots, v_s\}$ a set of vertices such that $G - S$ is a forest. We can think of all the graphs $G(A|B)$ with A, B disjoint subsets of S as vertices of the following tree, the root being $G = G(\emptyset|\emptyset)$



Filtrations

A path from the root to one of the leaves of the tree such that every graph that is the label of a vertex in the middle of the path has a non contractible independence complex is called a filtration of G



Back to Commutative Algebra

Let G be a ternary graph with non contractible independence complex and

$$\mathcal{F} : G_0, \ldots, G_s$$

a filtration of G.

Notation

1. The *vertex deletion number* of $\mathcal{F}$ is $\operatorname{del}(\mathcal{F}) = |\{i \mid G_i = G_{i-1} - v_i\}|$
2. The *deleted neighborhood* of $\mathcal{F}$ is $N(\mathcal{F}) = \{v_i \mid G_i = G_{i-1} - N[v_i]\}$
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Theorem

The independence complex of G is homotopy equivalent to $S^{\mathrm{depth}(\mathcal{F})-1}$.

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Theorem

- $\text{pd}(R/I(G)) = \text{del}(\mathcal{F}) + \sum_{v \in N(\mathcal{F})} \deg v$
- $\text{depth}(R/I(G)) = \text{depth}(\mathcal{F})$

In particular, the top betti number comes from the top monomial in the LCM lattice