Lefschetz properties and Rees algebras of squarefree monomial ideals

Thiago Holleben

Dalhousie University

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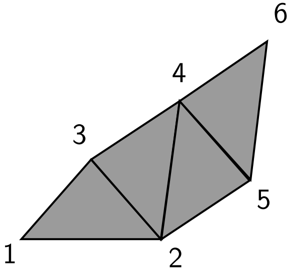
Stanley-Reisner, Facet (and incidence) ideals

A simplicial complex Δ on vertex set $[n]$ is a collection of subsets Δ of $[n]$ such that $\tau \subset \sigma \in \Delta \implies \tau \in \Delta$. We write $\Delta = \langle F_1, \ldots, F_s \rangle$ if $F_1, \ldots, F_s$ are the facets (maximal subsets) of Δ.



$$\Delta = \langle \{1, 2, 3\}, \{2, 3, 4\}, \{2, 4, 5\}, \{5, 4, 6\} \rangle$$

If we remove every 2-face of Δ (triangles), we get the complex $\Delta(1)$ which consists of the same vertices and edges of Δ, but no triangles

Stanley-Reisner, Facet (and incidence) ideals

Let $S = k[x_1, \ldots, x_n]$ and $\Delta = \langle F_1, \ldots, F_s \rangle$ a simplicial complex with vertex set $[n]$.

- The **Stanley-Reisner** ideal of Δ is the ideal

$$\mathcal{N}(\Delta) = (\prod_{i \in B} x_i : B \notin \Delta) \subset S$$

- The **Facet** ideal of Δ is the ideal

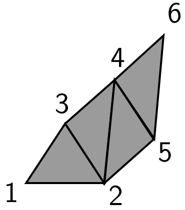
$$\mathcal{F}(\Delta) = (\prod_{i \in F_1} x_i, \ldots, \prod_{i \in F_s} x_i) \subset S$$

Both constructions give bijections between simplicial complexes and squarefree monomial ideals

Stanley-Reisner, Facet (and incidence) ideals

$$\mathcal{N}(\Delta) = (\prod_{i \in B} x_i : B \notin \Delta), \quad \mathcal{F}(\Delta) = (\prod_{i \in F_1} x_i, \dots, \prod_{i \in F_s} x_i)$$

$(x_1x_4, x_1x_5, x_3x_5, x_1x_6, x_2x_6, x_3x_6)$ $\longleftrightarrow$



$\mathcal{N}(\Delta)$

Δ

$\updownarrow$

$(x_1x_2x_3, x_2x_3x_4, x_2x_4x_5, x_4x_5x_6)$

$\mathcal{F}(\Delta)$

Lefschetz properties

Let I be a monomial ideal of $S = k[x_1, \dots, x_n]$ such that $A = S/I$ is artinian, and $L = x_1 + \cdots + x_n \in S_1$.

Definition

We say A satisfies the **weak Lefschetz property (WLP)** if the multiplication maps

$$\times L : A_i \to A_{i+1}$$

have full rank for every i.
If moreover the maps

$$\times L^j : A_i \to A_{i+j}$$

have full rank for every i, j, we say A satisfies the **strong Lefschetz property (SLP)**

A motivation from Combinatorics

Proposition

If A is an algebra that satisfies the WLP, then

$$\dim A_1 \leq \dim A_2 \leq \cdots \leq \dim A_k \geq \cdots \geq \dim A_d$$

for some k, in other words, the h-vector of A is **unimodal**.

We are particularly interested in algebras of the form:

$$A(\Delta) = \frac{S}{(\mathcal{N}(\Delta), x_1^2, \ldots, x_n^2)}$$

where Δ is a simplicial complex.

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$$A(\Delta) = \frac{S}{(\mathcal{N}(\Delta), x_1^2, \ldots, x_n^2)}$$

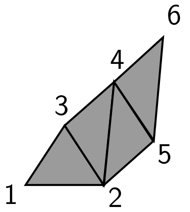
where Δ is a simplicial complex.

$$\dim A(\Delta)_i = f_{i-1} = \text{the number of } i-1 \text{ dimensional faces of } \Delta$$

An example with the SLP

$(x_1x_4, x_1x_5, x_3x_5, x_1x_6, x_2x_6, x_3x_6) \longleftrightarrow$

$\mathcal{N}(\Delta)$



Δ

The algebra

$$A(\Delta) = k[x_1, \ldots, x_6]/(\mathcal{N}(\Delta), x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2)$$

has the SLP whenever k is not a field of characteristic 2.

The bipartite property in Combinatorial Commutative Algebra

Let $I(G) = (x_i x_j \colon ij \text{ is an edge of } G)$ be the edge ideal of G

$$\begin{aligned}
\text{Not bipartite} &\iff \text{The rational map defined by } I(G) \text{ is birational} \\
&\iff I(G) \text{ is of linear type} \\
&\iff I(G)^{(m)} \neq I(G)^m \text{ for some } m \\
&\iff \text{Incidence matrix has full rank}
\end{aligned}$$

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But what can we say for simplicial complexes in general?

$\mathcal{F}(\Delta)$ Rees $\implies$ $\mathcal{N}(\Delta)$ Lefschetz

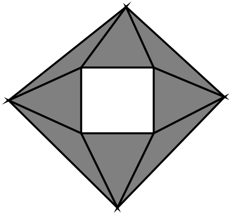
Theorem (-, 2024)

If Δ is connected and pure of dimension 2, then:

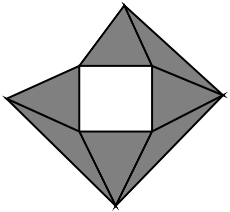
$$\mathcal{F}(\Delta) \text{ is of linear type } \implies A(\Delta) \text{ has the SLP}$$

Which properties of the Rees algebra of $\mathcal{F}(\Delta)$ can be translated into information on the Lefschetz properties of $\mathcal{N}(\Delta)$?

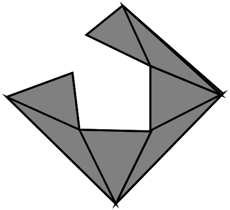
From linear type to Lefschetz properties: sufficient conditions visualized



Linear type results can't be used



Linear type results imply WLP in every odd characteristic



SLP in every odd characteristc

Symbolic powers

Symbolic powers of squarefree monomial ideals

Let $\mathcal{F}(\Delta) \subset S = k[x_1, \ldots, x_n]$ be a squarefree monomial ideal. The m-th symbolic power of $\mathcal{F}(\Delta)$ is:

$$\mathcal{F}(\Delta)^{(m)} = \bigcap_{P \in \mathrm{Ass}(\mathcal{F}(\Delta))} P^m$$

If $\mathcal{F}(\Delta) = (x_1x_2, x_2x_3, x_1x_3)$, then

$$\mathcal{F}(\Delta)^{(2)} = (x_1x_2x_3, x_1^2x_2^2, x_2^2x_3^2, x_1^2x_3^2) \neq \mathcal{F}(\Delta)^2$$

Symbolic Powers and Lefschetz properties are not compatible

Theorem (-, 2024)

Let Δ be a pure simplicial complex with at least as many facets as vertices.

- *If $\mathcal{F}(\Delta)^{(m)} = \mathcal{F}(\Delta)^m$ for all m, then $A(\Delta)$ fails the SLP.*

Corollary (-, 2024)

Let G be a bipartite graph with $n \geq 5$ vertices and $w(G)$ the whiskered graph. Let

$$I(w(G)) = (x_{i_{1,1}}, \dots, x_{i_{1,n}}) \bigcap \cdots \bigcap (x_{i_{r,1}}, \dots, x_{i_{r,n}})$$

and $\Delta = \langle \{i_{1,1}, \dots, i_{1,n}\}, \dots, \{i_{r,1}, \dots, i_{r,n}\} \rangle$. Then $A(\Delta)$ fails the SLP.

The symbolic defect: a horizontal perspective

Symbolic Defect sequence of an ideal (GGSVT, 2018)

Let I be an ideal, define

$$\text{sdefect}(I, m) = \text{ the minimal number of generators of } I^{(m)}/I^m$$

for every m.

Theorem (GGSVT, 2018)

If I is the ideal generated by every squarefree monomial ideal of degree d in n variables, then

$$\text{sdefect}(I, 2) = \binom{n}{d+1}$$

In other words, $\text{sdefect}(I, 2)$ *is the number of d-faces of the simplex on n vertices.*

Symbolic defect polynomials

$$\mathrm{sdefect}(\mathcal{F}(\Delta), m)$$

$$\vdots$$

$$\mathrm{sdefect}(\mathcal{F}(\Delta), 4)$$

$$\mathrm{sdefect}(\mathcal{F}(\Delta), 3)$$

$$\mathrm{sdefect}(\mathcal{F}(\Delta), 2)$$

Symbolic defect polynomials

$$
\begin{array}{cccc}
\operatorname{sdefect}(\mathcal{F}(\Delta(1), m) & \cdots & \operatorname{sdefect}(\mathcal{F}(\Delta(d-1), m) & \operatorname{sdefect}(\mathcal{F}(\Delta), m) \\
\vdots & \ddots & \vdots & \vdots \\
\operatorname{sdefect}(\mathcal{F}(\Delta(1), 4) & \cdots & \operatorname{sdefect}(\mathcal{F}(\Delta(d-1), 4) & \operatorname{sdefect}(\mathcal{F}(\Delta), 4) \\
\operatorname{sdefect}(\mathcal{F}(\Delta(1), 3) & \cdots & \operatorname{sdefect}(\mathcal{F}(\Delta(d-1), 3) & \operatorname{sdefect}(\mathcal{F}(\Delta), 3) \\
\operatorname{sdefect}(\mathcal{F}(\Delta(1), 2) & \cdots & \operatorname{sdefect}(\mathcal{F}(\Delta(d-1), 2) & \operatorname{sdefect}(\mathcal{F}(\Delta), 2)
\end{array}
$$

Symbolic defect polynomials

The second symbolic defect polynomial

The **second symbolic defect polynomial** of a pure simplicial complex Δ is:

$$\mu(\Delta, 2, x) = \sum_i \mathrm{sdefect}(\mathcal{F}(\Delta(i)), 2) x^{i+2}$$

Symbolic defect polynomials

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Theorem (-, 2024)

Let Δ be a flag simplicial complex.

- *The coefficient of x^3 in $\mu(\Delta, 2, x)$ is equal to the number of triangles of Δ.*
- *The sequence of coefficients of $\mu(\Delta, 2, x)$ has no internal zeros.*

A couple of examples

Let $\mathcal{N}(\Delta) = (x_i x_{i+1} : 1 \leq i \leq 14) \subset k[x_1, \ldots, x_{15}]$. Then

$$\mu(\Delta, 2, x) = 286x^3 + 495x^4 + 462x^5 + 210x^6 + 36x^7 + x^8$$

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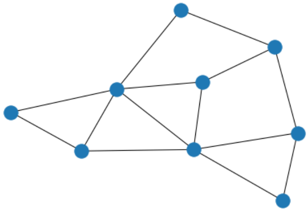
$$\mu(\Delta, 2, x) = 286x^3 + 495x^4 + 462x^5 + 210x^6 + 36x^7 + x^8$$

and the f-vector of Δ is:

$$(1, 15, 91, 286, 495, 462, 210, 36, 1)$$

A couple of examples



The Stanley-Reisner complex Δ of the edge ideal of the graph above has

- $\mu(\Delta, 2, x) = 17x^3 + 5x^4$
- f-vector: $(1, 9, 22, 17, 4)$

So the two are not always the same

Unimodality? Log-concavity? f-vectors?

Questions

- When is the second symbolic defect polynomial of a complex equal to its f-vector?
- When is the second symbolic defect polynomial of a complex unimodal?

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Do the questions above hold when Δ is the independence complex of a forest?