Roller coasters and Hilbert series of Artinian Gorenstein Algebras

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Summary
The search for a characterization of h-vectors of Artinian Gorenstein (AG) algebras is a very hard problem, special cases of this problem have been studied extensively in other areas. In this poster, we show the existence of AG algebras breaking several inequalities that one could maybe expect them to satisfy. This is joint work with Susan Cooper, Sara Faridi, Lisa Nicklasson and Adam Van Tuyl.

The question of characterizing Gorenstein h-vectors

Let $h_0, h_1, \dots, h_d$ be a sequence of positive integers. What do we need to impose to guarantee that there exists an Artinian Gorenstein (AG) algebra with $\dim A_i = h_i$?
Some necessary conditions are:

1. $h_0 = h_d = 1$
2. $h_i = h_{d-i}$
3. The inequalities from Macaulay's theorem are satisfied

Although the question as stated is open, there are several similar questions in Combinatorics, where a lot of progress has been made:

1. A complete characterization of h-vectors of simplicial spheres is known (g-theorem)
2. Hilbert series of quadratic level Artinian monomial algebras are known to be increasing up to the first half, and when $h_1 = 2d$, the last third of the sequence is decreasing.

In this poster, we explore techniques used in Combinatorics to prove 2. above in order to prove the existence of "bad AG algebras".

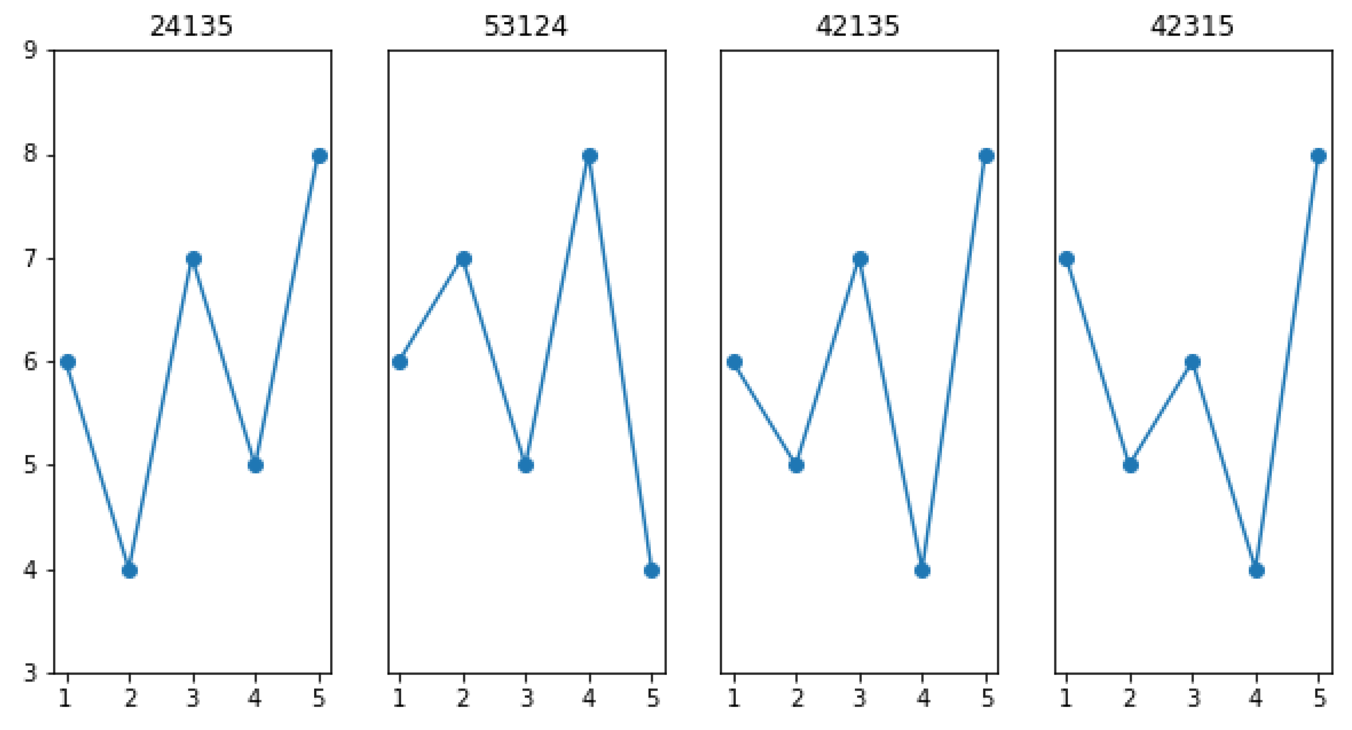
The beginning of the story (from a different perspective)

In 1987, Alavi, Malde, Schwenk and Erdős showed the following:

Definition 1. A family of sequences H is said to be **unconstrained** if for every n and every permutation of $[n]$, there exists a sequence $\{h_1, \dots, h_n\} \in H$ such that

$$h_{\pi(1)} < h_{\pi(2)} < \dots < h_{\pi(n)}.$$

Theorem 2. The family of Hilbert series of quadratic Artinian monomial algebras is unconstrained.



In 2000, Brown, Dilcher and Nowakowski conjectured the following:

Definition 3. An Artinian algebra A of socle degree d is **level** if

$$A_d = \{f \in A \colon fA_1 = 0\}$$

Conjecture 4. The Hilbert series of a level quadratic Artinian monomial algebra is unimodal.

One year later Michael and Traves found counter-examples to Conjecture 4 and conjectured the following result which is now a theorem proved in 2017 by Cutler and Pebody:

Theorem 5 (Roller-Coaster Theorem). The Hilbert series of quadratic level monomial algebras is increasing up to the middle. After the middle, this family of sequences is unconstrained.

The Gorenstein case can be very bad

Definition 6. A sequence $h_1, \dots, h_d$ is **unimodal** if there exists k such that $h_1 \leq \dots \leq h_k \geq \dots \geq h_d$. If a sequence is not unimodal, it has a **valley** at h_k if $h_{k-1} > h_k < h_{k+1}$

Already in 1978 Stanley gave an example of a nonunimodal Gorenstein h-vector:

$$1, 13, 12, 13, 1$$

In 1995, Boij showed the following:

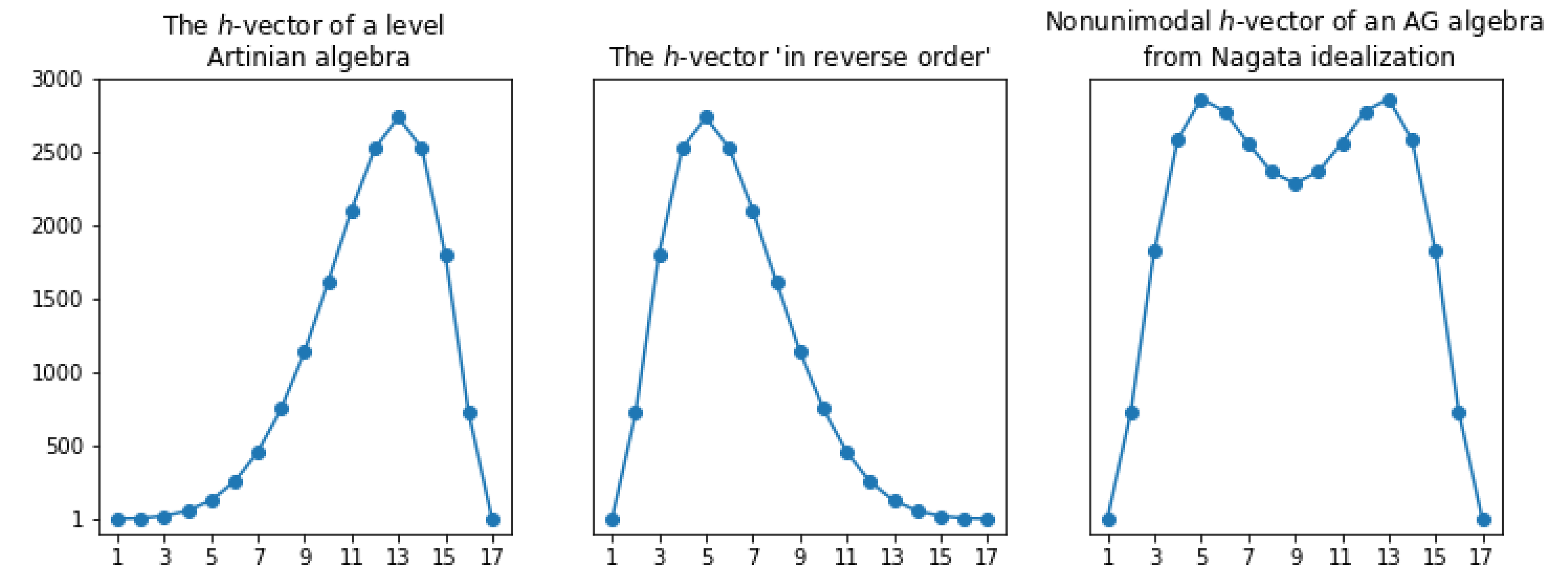
Theorem 7. The h-vectors of Artinian Gorenstein algebras can have arbitrarily many valleys.

The "trick": Nagata idealization

One common technique to generate bad Artinian Gorenstein algebras from level Artinian algebras is the following:

Lemma 8 (Nagata Idealization). Let A be a level Artinian algebra of socle degree d and h-vector $h_0, h_1, \dots, h_{d-1}, h_d$. There exists an Artinian Gorenstein algebra of socle degree $d+1$ with h-vector:

$$h_0, h_1 + h_d, h_2 + h_{d-1}, \dots, h_{d-1} + h_2, h_d + h_1, h_0$$



The main result: A roller coaster theorem for AG algebras

Using the techniques by Cutler and Pebody + Nagata idealization, we are able to show the following:

Theorem 9 (A Roller Coaster theorem for AG algebras - CFHNVT, 24+). The first half of h-vectors of Artinian Gorenstein algebras is an unconstrained sequence. In other words, for every n and every permutation π of $\{1, \dots, \lfloor \frac{n}{2} \rfloor\}$ there exists an Artinian Gorenstein algebra A with Hilbert series $h_0, h_1, \dots, h_n$ such that

$$h_{\pi(1)} < h_{\pi(2)} < \dots < h_{\pi(\lfloor \frac{n}{2} \rfloor)}.$$

Now the question becomes:

How far can we push this result? What other properties can we assume these algebras have, while the Roller Coaster theorem still holds?

Perazzo forms and Lefschetz properties

In the 1850s, Hesse claimed that every hypersurface $X \subset \mathbb{P}^N$ with vanishing hessian is a cone. This claim was then disproved by P. Gordan and M. Noether who showed that being a cone means the partial derivatives of the defining form f are **linearly dependent**, while f has vanishing hessian if and only if its partial derivatives are **algebraically dependent**. A homogeneous polynomial $f(x_1, \dots, x_s, u_1, \dots, u_n)$ is a **Perazzo form** if it can be written as

$$f = x_1 g_1(u_1, \dots, u_n) + \dots + x_s g_s(u_1, \dots, u_n) + g(u_1, \dots, u_n)$$

where all the g_i are algebraically dependent. Hypersurfaces coming from Perazzo forms are examples of hypersurfaces with vanishing hessian that are not cones.

Definition 10 (Lefschetz properties). An Artinian algebra A is said to have the weak Lefschetz property (WLP) if for a general linear form ℓ, the multiplication maps $\ell \times A_i \to A_{i+1}$ have full rank for every i. The algebra A is said to have the strong Lefschetz property (SLP) if the multiplication maps $\ell^j : A_i \to A_{i+j}$ have full rank for every i, j.

Using the theory of Macaulay duality, it is possible to associate an AG algebra to a homogeneous form f. Due to results of Watanabe and Maeno, it is known that the algebras dual to Perazzo forms fail the SLP, but examples have been found where they have the WLP, and where they fail the WLP.

In this direction, we show the following:

Theorem 11 (A Roller Coaster theorem for Perazzo forms - CFHNVT, 24+). The first half of the Hilbert functions of Perazzo forms is unconstrained.

Final remarks

- The methods we use come from Graph theory, and in particular, proofs are ε, δ proofs that show **existence** but do not give us examples. Methods for generating such algebras would also be of interest to graph theorists.
- Since we are dealing with quadratic monomial ideals, it is natural to ask whether there is a Roller-Coaster theorem for Koszul AG algebras, or even AG algebras with a quadratic Gröbner basis. D'ali and Venturello characterized which Nagata idealizations of quadratic level monomial ideals have these properties. We are currently looking into this.
- It is a known drawback of the Nagata idealization trick that the codimension of the algebras generated are extremely high. An interesting question becomes what is the minimal codimension of an AG algebra satisfying the inequalities for a given permutation π.
- It is important to note that there are at least three perspectives one can take to study the behavior of h-vectors of AG algebras. The interesting consequence is that conjectures made from different perspectives point to different behaviors. Since Nagata idealization can be used to generate "bad AG algebras" from "bad quadratic level monomial algebras", it becomes interesting to see how far one can push this strategy.
- Using different techniques (also from graph theory) we are able to construct **explicit** Perazzo forms that fail WLP in arbitrarily many degrees. The algebras associated to our examples are Koszul and have a quadratic Gröbner basis.