Positivity through analytic spread

Thiago Holleben
Dalhousie University
hollebenthiago@dal.ca

Summary
The Rees Algebra of an ideal is an important object in Commutative Algebra that contains information about several invariants. In particular, when I is either an equigenerated monomial ideal or the jacobian ideal of a product of linear forms, the positivity of some invariants can be determined by the rank of a matrix. We use the theory of Lefschetz properties to define polynomials with nonnegative coefficients associated to combinatorial objects. We then give examples where these polynomials become well known polynomials in Combinatorics.

Introduction

It is a well known phenomena that many multiplicity theories have connections to volumes of convex bodies. When I is an equigenerated monomial ideal, the ε, j and mixed multiplicities can be computed as mixed volumes of convex bodies or regions in $\mathbb{R}^n$. In particular, to determine whether these multiplicities are positive or not, it is only necessary to compute the rank of a specific matrix associated to the convex body.

The remark above is a very specific case of well known results relating positivity of multiplicities and analytic spread being maximal.

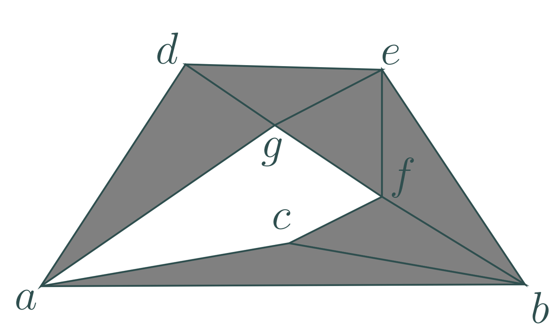
When I is an equigenerated squarefree monomial ideal, the same matrix that is used to determine whether multiplicities are positive can be used as a sufficient condition for symbolic powers to not be equal to ordinary powers.

The theory of Lefschetz properties has many connections to Combinatorics, Geometry and Topology and its main focus is to show that some linear maps arising as multiplication maps in Artinian algebras have full rank.

Here, we apply results from Lefschetz properties to define polynomials made out of algebraic invariants that have positive coefficients.

Pure O-sequences and pure simplicial complexes

Definition 1. A simplicial complex Δ on vertex set V is a collection of subsets of V such that if $\sigma \in \Delta$ and $\tau \subset \sigma$, then $\tau \in \Delta$. Elements in Δ are called its faces, maximal elements are called facets. We say Δ is pure if every facet has the same size. If the maximum size of a face of Δ is i, we say Δ is $i-1$ dimensional.



A pure simplicial complex Δ with facets $abc, bcf, bfe, fge, dge, adg$

Theorem 2 (Hausel, [4])**.** Let A be an Artinian level algebra of socle degree k over a field of characteristic zero defined by a monomial ideal and L a general linear form. Then the following maps are injective

$$\times L^{k-2i} : A_i \to A_{k-i} \quad \text{for } 2i < k$$

We apply the result above to show the following.

Theorem 3. Let Δ be a pure simplicial complex with maximal facets of size d on vertex set V. Then the analytic spread of the ideals generated by faces of size i of Δ for $1 \leq i \leq d-1$ is equal to $|V|$. In particular, these ideals have maximal analytic spread.

Symbolic powers and f-vectors

Let I be a squarefree monomial ideal in $\Bbbk[x_1, \dots, x_n]$ and Δ its Stanley-Reisner complex. The m-th symbolic power of I can be computed as

$$I^{(m)} = \bigcap_{F \subset \Delta} (x_i : i \notin F)^m$$

where F runs over facets of Δ. It is known (see [7]) that if the analytic spread of I is n, there exists t such that

$$I^{(t)} \neq I^t.$$

Corollary 4. Let Δ be a pure simplicial complex of dimension d and I_i the ideal generated by the faces of dimension i of Δ. Then $I_i^{(2)} \neq I_i^2$ for every $1 \leq i \leq d-1$.

In order to measure how different the symbolic powers of an ideal are from ordinary powers, in [3] Galetto et al. defined the **symbolic defect** of an ideal as

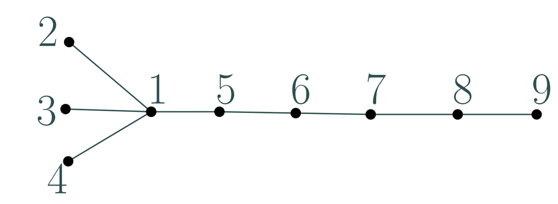
$$\operatorname{sdefect}(I, m) = \mu(I^{(m)}/I^m) \quad \text{where } \mu(-) \text{ denotes the minimal number of generators.}$$

Definition 5. Due to the corollary above we define the **m-th symbolic defect polynomial** of a simplicial complex Δ of dimension d as

$$\mu_m(\Delta, t) = \operatorname{sdefect}(I_1, 2) + \operatorname{sdefect}(I_2, 2)t + \cdots + \operatorname{sdefect}(I_{d-1}, 2)t^{d-2}$$

Corollary 6. The second symbolic defect polynomial of Δ has nonzero coefficients. Moreover, the number of $i+1$-faces of Δ is a lower bound for $\operatorname{sdefect}(I_i, 2)$.

Example 7. Let G be the following graph.



Then the second symbolic defect polynomial of the Stanley-Reisner complex Δ of G is

$$\mu_2(\Delta, t) = 38 + 26t + 9t^2 + 1t^3$$

and the f-vector of Δ is $(1, 9, 28, 38, 26, 9, 1)$.

Question 8. Let Δ be the Stanley-Reisner complex of a forest. Are the coefficients of $\mu_2(\Delta, t)$ equal to the face numbers of Δ?

Multiplicities and Eulerian numbers

The convex hull of lattice points that consist of k ones and $n-k$ zeros in $\mathbb{R}^n$ is called the n, k-hypersimplex. In Combinatorics, Eulerian numbers denoted by $A(n,k)$ count the numbers of permutations of n elements that have exactly k ascents. A standard fact says the volume of the n, k-hypersimplex is equal to $A(n,k)$.

In [1], Alilooee et al. noticed that mixed multiplicities of squarefree monomial ideals generated by all squarefree monomials of degree k correspond to $A(n,k)$ and the more general mixed Eulerian numbers. Notice that these ideals can be seen as ideals generated by the i-faces of a simplex, for $i = 0, \dots, d$. When Δ is a pure simplicial complex, due to Theorem 3 and the results in [2, 8] we show the following.

Corollary 9. Let Δ be a pure simplicial complex of dimension d in n vertices. For $i = 0, \dots, d-1$, let I_i denote the ideal generated by the i faces of Δ. Then

$$e_{(c_0, \dots, c_{d-1})}(I_0 | I_1, \dots, I_{d-1}) > 0 \quad \text{for every } c_0 + \cdots + c_{d-1} = n-1$$

where the numbers above are the mixed multiplicities of $I_0, \dots, I_{d-1}$.

Similarly, j and ε multiplicities of the ideals I_i defined above are also positive.

The nonmonomial case

In [5], June Huh showed that mixed multiplicities of Jacobian ideals of products of linear forms are equal to coefficients of a combinatorial polynomial known as the characteristic polynomial of a matroid. In particular, we may compute the analytic spread of jacobian ideals of products of linear forms as follows.

Example 10. Consider $h = xyz(x+z) \in \Bbbk[x, y, z]$. Let M be the following matrix:

$$\begin{array}{c} \\ x \\ y \\ z \\ x+z \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \end{array}$$

Then the analytic spread of the Jacobian ideal of h is the rank of M.

Definition 11. Let A be an artinian graded algebra of socle degree d over a field of characteristic zero satisfying the Strong Lefschetz Property and L a linear form that is a Lefschetz element.

Then to each map $\times L^i : A_1 \to A_{i+1}$, there is a product of linear forms h_i as defined above. We define the **linear skeleton** of the pair (A, L) as the sequence of jacobian ideals

$$(J_{h_1}, \dots, J_{h_d}).$$

Similarly to Theorem 3, every pair (A, L) gives rise to many sequences of positive integers.

References

[1] Ali Alilooee, Ivan Soprunov, and Javid Validashti. Generalized multiplicities of edge ideals. *J. Algebraic Combin.*, 47(3):441–472, 2018.

[2] Federico Castillo, Yairon Cid-Ruiz, Binglin Li, Jonathan Montaño, and Naizhen Zhang. When are multidegrees positive? *Sém. Lothar. Combin.*, 85B:Art. 46, 12, 2021.

[3] Federico Galetto, Anthony V. Geramita, Yong-Su Shin, and Adam Van Tuyl. The symbolic defect of an ideal. *J. Pure Appl. Algebra*, 223(6):2709–2731, 2019.

[4] Tamás Hausel. Quaternionic geometry of matroids. *Cent. Eur. J. Math.*, 3(1):26–38, 2005.

[5] June Huh. Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs. *J. Amer. Math. Soc.*, 25(3):907–927, 2012.

[6] Jack Jeffries and Jonathan Montaño. The j-multiplicity of monomial ideals. *Math. Res. Lett.*, 20(4):729–744, 2013.

[7] Susan Morey and Rafael H. Villarreal. Edge ideals: algebraic and combinatorial properties. In *Progress in commutative algebra 1*, pages 85–126. de Gruyter, Berlin, 2012.

[8] Ngo Viet Trung and Jugal Verma. Mixed multiplicities of ideals versus mixed volumes of polytopes. *Trans. Amer. Math. Soc.*, 359(10):4711–4727, 2007.