Coinvariant stresses, Lefschetz properties and random complexes

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Summary

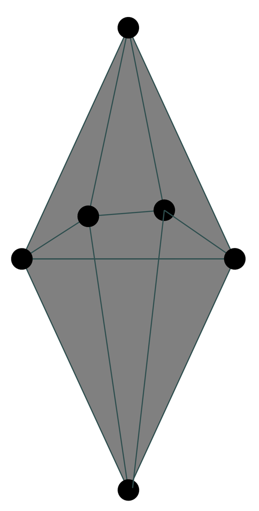
The g-conjecture is one of the most important questions in the study of f, h and g-vectors of simplicial complexes. Two very successful approaches to study it are through Lefschetz properties and rigidity theory. These two strategies are connected via the algebraic notion of inverse systems, where the main idea that bridges the two perspectives is to translate vertex coordinates of a geometric complex, to a set of linear forms (a linear system of parameters) of the Stanley-Reisner ideal of the abstract complex. Since the linear forms have to be very carefully chosen, computation in this setting can become overwhelming. In this poster we present the idea of using elementary symmetric polynomials instead of linear forms. As a consequence, we generalize a result on Lefschetz properties of monomial almost complete intersections of Migliore, Miró-Roig and Nagel.

f, h and g-vectors

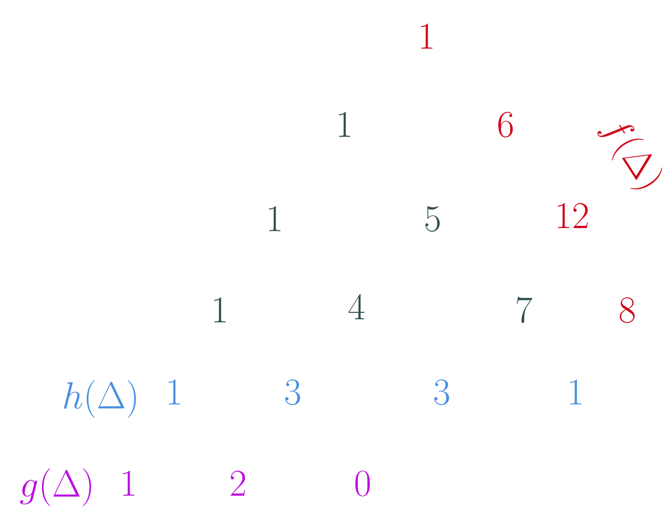
A **simplicial complex** Δ is a collection of subsets of $[n]$ such that

$$\sigma \in \Delta \quad \text{and} \quad \tau \subset \sigma \implies \tau \in \Delta$$

Let $f_i = |\{\sigma \in \Delta \colon |\sigma| = i+1\}|$ denote the number of i-faces of Δ, and the vector $f(\Delta) = (f_{-1}, \dots, f_d)$ the **f-vector of** Δ, where d is called the **dimension** of Δ. It turns out that the sequences of numbers that contain information on the combinatorics/topology of Δ in a more useful form are the h and g-vectors of Δ, which we define below in an example.



$\Delta \quad f_0 = 6,\ f_1 = 12,\ f_2 = 8$



The complex Δ above is homeomorphic to a 2-sphere. Such complexes are called **simplicial spheres**.

The symmetry of $h(\Delta)$ and the entries of $g(\Delta)$ being nonnegative are not a coincidence. In 1971 McMullen conjectured the following (originally for simplicial polytopes)

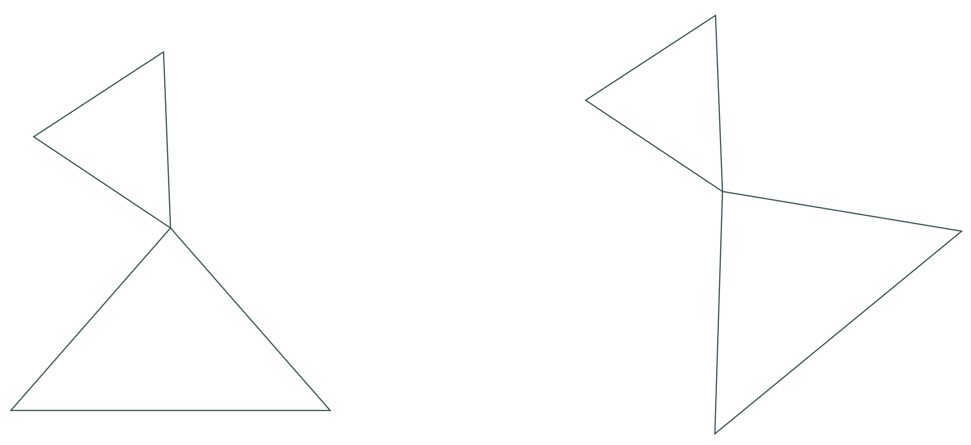
Theorem 1 (**g-theorem for simplicial spheres**, S-80, BL-81, A-18, PP-20, APP-21)**.** A sequence of numbers $(h_0, \dots, h_d)$ is the h vector of a $(d-1)$–dimensional simplicial sphere if and only if the following hold:

1. $h_0 = 1$
2. $h_i = h_{d-i}$ (Gorensteiness/Poincaré duality/Dehn-Sommerville relations)
3. There exists a standard graded artinian algebra A such that $\dim A_i = g_i$

Rigidity theory, stresses and motions: an unexpected approach to positivity

A direct corollary of 3. in the g-theorem is that the numbers g_i are all nonnegative. In 1987, Kalai gave a different proof that g_2 is always nonnegative using tools from **rigidity theory**.

A **bar-and-joint framework** is a tuple $P = (G, p)$ where $G = (V, E)$ is a graph and $p : V(G) \to \mathbb{R}^n$ a choice of coordinates for the vertices of G (an embedding). In our context, the embedding will always be **generic**. A framework is called **flexible** if it admits a nontrivial motion that preserves edge lengths



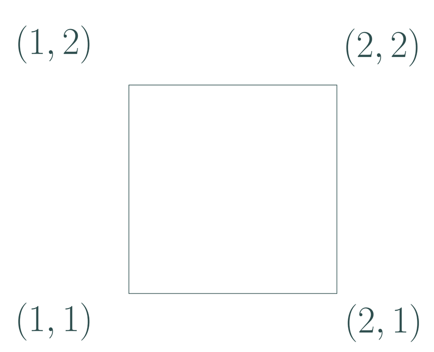
A nontrivial motion

It is known that studying rigidity of a (generic) framework is equivalent to computing the kernel of a matrix called the **rigidity matrix** $M(G, p)$. Elements in the kernel of $M(G, p)^T$ are called **stresses** of the framework (G, p). Kalai showed that g_2 is always equal to the dimension of the space of stresses, and hence must be positive.

Lee's amazing idea: Partial Differential Equations!

After Kalai's proof that $g_2 \geq 0$ using rigidity theory, it became an interesting problem to try to generalize the notion of stress to arbitrary dimensions. Such a generalization had the potential to be used in a full proof of the g-conjecture.

In 1996, Lee noticed that computing stresses was equivalent to solving systems of partial differential equations arising from the non-faces of a simplicial complex and the embedding. An example of Lee's idea can be seen below



$$\begin{aligned}\theta_1 &= x_1 + 2x_2 + 2x_3 + x_4 \\ \theta_2 &= 2x_1 + 2x_2 + x_3 + x_4\end{aligned}$$

$$\begin{cases} f_{x_1} + 2f_{x_2} + 2f_{x_3} + f_{x_4} = 0 \\ 2f_{x_1} + 2f_{x_2} + f_{x_3} + f_{x_4} = 0 \\ f_{x_1x_3} = 0 \\ f_{x_2x_4} = 0 \end{cases}$$

$\{1,3\}$ and $\{2,4\}$ are the mininmal nonfaces of the square

Lee noticed that when f is a solution of degree 2 to the system above, the coefficients of f determine a stress of the framework and vice-versa. This observation led to Lee's definition of affine and linear stresses of a Cohen-Macaulay complex.

Definition 2 (Lee-1996)**.** A **k-linear stress** of a d-dimensional simplicial complex Δ is a solution of degree k to a system of PDEs as above, where the equations come from $d+1$ general linear forms, and the minimal nonfaces of Δ.

A **k-affine stress** of a d-dimensional simplicial complex Δ is a solution of degree k to a system of PDEs as above, where the equations come from $d+1$ general linear forms, the minimal nonfaces of Δ and an extra linear PDE given by the sum of the partial derivatives.

General linear forms make it messy — go coinvariant!

Stanley's proof of the g-theorem for simplicial polytopes uses results from algebraic geometry, and in that case, the linear forms can be chosen based on the structure of the simplicial polytope.

When it comes to simplicial spheres though, the lack of underlying geometry leads to (very!) generic choices of linear forms. As a consequence, computations can get overwhelming and become sensitive to coefficients

It turns out that the algebraic theory of inverse systems – the algebra behind the PDEs – does not require the PDEs to be linear, only homogeneous. The key idea then is to use a set of homogeneous forms arising from combinatorics, to try the same techniques

Theorem 3. Let Δ be a d-dimensional simplicial complex with Stanley-Reisner ideal I_Δ. Then the elementary symmetric polynomials $e_1, \dots, e_{d+1}$ form a linear system of parameters of I_Δ.

In other words, we can replace the linear PDEs from before by PDEs of the form

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{x_{i_1} \cdots x_{i_k}} = 0$$

Definition 4 (**Coinvariant stresses, - (2025)**)**.** Let Δ be a d-dimensional Cohen-Macaulay complex. A **coinvariant stress** of Δ of degree k is a solution to the system of PDEs:

$$\begin{cases} \displaystyle\sum_{1 \leq i_1 < \cdots < i_s \leq n} f_{x_{i_1} \cdots x_{i_s}} = 0 & \text{for } s \leq d+1 \\ f_{x_{i_1} \cdots x_{i_j}} & \text{where } \{i_1, \dots, i_j\} \notin \Delta \end{cases}$$

Theorem 5 (**Top coinvariant stresses and top homology**, - (2025))**.** Let Δ be a d-dimensional simplicial complex such that $c_1F_1 + \cdots + c_sF_s \in \tilde{H}_d(\Delta; \Bbbk)$. Then

$$c_1x_{F_1}V(F_1) + \cdots + c_sx_{F_s}V(F_s)$$

is a coinvariant stress of Δ of degree $\binom{d+2}{2}$, where $x_{F_i} = \prod_{j \in F_i} x_j$ and $V(F_i) = \prod_{1 \leq a < b \leq n} (x_a - x_b)$

When Δ is the boundary of the simplex, Theorem 5 recovers the classical fact that the unique nonzero polynomial in the coinvariant ring of the symmetric group is the Vandermonde determinant.

Failure should be expected (a counterintuitive application)

The key point of studying these systems of PDEs from the point of view of the g-theorem, is that the following are equivalent:

1. Computing the dimensions of solution spaces to the PDE system + one linear form in degree k
2. Proving condition 3. holds in degree k

This equivalence can be stated in algebraic terms as saying that a specific algebra satisfies the **weak/strong Lefschetz property (WLP/SLP)**. More specifically, a standard graded algebra B satisfies the WLP if there exists a linear form $L \in B_1$ such that the multiplication maps $\times L : B_i \to B_{i+1}$ have full rank for all i.

A consequence of Theorem 5 is the following:

Theorem 6 (**Why can failure happen?** - (2025))**.** Let Δ be a d-dimensional complex such that $d > 0$, $f_{d-1} \geq f_d$ and $\tilde{H}_d(\Delta; \Bbbk) \neq 0$. Then the algebra

$$\frac{\Bbbk[x_1, \dots, x_n]}{I_\Delta + (x_1^{d+2}, \dots, x_n^{d+2})} \quad \text{fails the WLP.}$$

When Theorem 6 is applied to the boundary of a simplex, we recover a famous result of Migliore, Miró-Roig and Nagel on the failure of the WLP of monomial almost complete intersections.

The assumptions in Theorem 6 are actually all very mild:

1. It is known that under generalized Erdős–Rényi models, the probability that a complex has nonzero top homology converges to 1 assuming the probability parameter is not too low.
2. Under generalized Erdős–Rényi models if the probability parameters are not too high, the probability that $f_{d-1} \geq f_d$ also converges to 1 as the number of vertices increases.

Theorem 7 (**"Everything" fails** - (2025))**.** For every $d > 0$ there exists a nonempty interval $(c_d, d+1)$ such that if $c \in (c_d, d+1)$ and $p = \frac{c}{n}$ is the coin-flip parameter in a generalized Erdős–Rényi model, then

$$\lim_{n \to \infty} \mathbb{P}\left(\frac{\Bbbk[x_1, \dots, x_n]}{I_\Delta + (x_1^{d+2}, \dots, x_n^{d+2})} \quad \text{fails the WLP} \right) = 1.$$

Further work

Question 8 (**A coinvariant algebraic g-conjecture**)**.** Let Δ be a d-dimensional $\Bbbk$-homology sphere, I_Δ be the Stanley-Reisner ideal of Δ and e_i the i-th elementary symmetric polynomial. Does the ring

$$\frac{\Bbbk[x_1, \dots, x_n]}{I_\Delta + (e_1, \dots, e_{d+1})}$$

satisfy the SLP?

Check the preprint on arxiv!