Lefschetz Properties and Mixed Multiplicities

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Summary

Lefschetz Properties have been studied from many different perspectives, and recently, connections to different areas of Mathematics have been shown, including Algebraic Geometry, Differential Geometry and Combinatorics. Mixed multiplicities of ideals have also been used recently to describe invariants from different areas of Mathematics such as Convex Geometry, Topology and Combinatorics. We apply Huh's results on the characteristic polynomial of hyperplane arrangements and the positivity results of mixed multiplicities by Trung, to describe the Weak Lefschetz Properties of artinian algebras that are the quotient of a monomial ideal over a field of characteristic zero.

The Weak Lefschetz Property (WLP)

Let I be a monomial ideal in $k[x_1, \dots, x_n]$ where k is a field of characteristic zero and assume $A = R/I$ is artinian.

Definition 1 (WLP for monomial ideals)**.** Let $L = x_1 + \cdots + x_n$. We say the algebra A has the **Weak Lefschetz Property**(WLP) in degree i if the multiplication map $\times L : A_i \to A_{i+1}$ has full rank.

Example 2. Let $R = k[x, y, z]$, $I = (x^3, y^3, z^3, xy, yz)$ and $A = R/I$. To check if A has the WLP in degree 1 we need to check if the following matrix has full rank:

$$\begin{array}{c} \\ x^2 \\ y^2 \\ z^2 \\ xz \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \end{array}$$

One can think of the matrices that represent the maps $\times L : A_i \to A_{i+1}$ as "divisibility matrices": the ij-th entry is nonzero if and only if the monomial associated to the i-th row is divisible by the monomial associated to the j-th column, and in that case, the entry is 1.

A nice ambient space

In order to apply Huh's result, we need a polynomial ring and an ideal that contain the combinatorial structure of A. To do that, we first define the polynomial ring:

Definition 3. Let A be the quotient of a polynomial ring R in n variables over k by a monomial ideal I. We write R_I for the polynomial ring:

$$R_I = \mathbb{C}[t_m | m \text{ is a monomial} \notin I]$$

To define the ideal, first note that we can associate a product of linear forms h to the matrix $\times L : A_i \to A_{i+1}$ as follows:

Example 4. Let M be the matrix from Example 2. Then taking the columns of M to be the linear coefficients of linear forms in R_I we have:

$$h = \underbrace{(t_{x^2} + t_{xz})}_{x} \underbrace{(t_{y^2})}_{y} \underbrace{(t_{z^2} + t_{xz})}_{z}$$

that is, h is the product of the linear forms $l_m = \sum_{m|m'} t_{m'}$ for every m in A_1 (each column).

Hyperplane arrangements

Definition 5. A **hyperplane arrangement** is a collection $\mathcal{A}$ of hyperplanes in $\mathbb{C}^n$. The **rank** of $\mathcal{A}$ is the dimension of the vector space spanned by the normal vectors of the hyperplanes in $\mathcal{A}$. Since each hyperplane B in $\mathcal{A}$ is the set of zeros of a linear form l_B, we say the polynomial

$$h = \prod_{B \in \mathcal{A}} l_B$$

is the **defining polynomial** of $\mathcal{A}$. Note that a product of linear forms defines a hyperplane arrangement for the same reason.

We call the ideal J_h generated by the partial derivatives of h the **jacobian ideal** of h.

Example 6. Let $\mathcal{A}$ be the hyperplane arrangement defined by

$$(a + d)(b)(c + d) \in \mathbb{C}[a, b, c, d]$$

then since the 3 normal vectors are linearly independent the rank of the arrangement is 3.

More specifically, to compute the rank of $\mathcal{A}$ we need to compute the column rank of the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Mixed Multiplicities

Let $R = k[x_1, \dots, x_n]$, $\mathfrak{m} = (x_1, \dots, x_n)$ and J an arbitrary ideal of R. Every graded component of the algebra

$$R(\mathfrak{m}|J) := \bigoplus_{(u,v) \in \mathbb{N}^2} \mathfrak{m}^u J^v / \mathfrak{m}^{u+1} J^v$$

has finite length. The numerical function that maps u, v to the length of $R(\mathfrak{m}|J)_{uv}$ is a polynomial (for $u, v \gg 0$) of the form:

$$P(u, v) = \sum_{i=0}^{n} e_{(n-i,i)}(\mathfrak{m}|J) u^{n-i} v^i + \text{ terms of lower degree}$$

The nonnegative numbers $e_{(n-i,i)}(\mathfrak{m}|J)$ are called the **mixed multiplicities** of $\mathfrak{m}$ and J. Since $P(u, v)$ is a polynomial in more than one variable, many basic properties that hold for polynomials in one variable are not true for $P(u, v)$. The one that is important to our setup is the fact that the coefficients of monomials of maximal degree are nonzero. This leads to the question of when is a mixed multiplicity positive, which was answered by Trung in [8]:

Theorem 7 (Trung, [8])**.**

$$e_{(n-i,i)}(\mathfrak{m}|J) > 0 \iff 0 \leq i \leq \ell(J) - 1,$$

where $\ell(J)$ is the analytic spread of J

Rank and analytic spread

In [2], Huh showed that, after taking a convolution, the mixed multiplicities of the jacobian ideal of a hyperplane arrangement coincide with the coefficients of a polynomial that is an important combinatorial invariant of an arrangement called the **characteristic polynomial**. Combining this result with Theorem 7 one concludes:

Theorem 8. The analytic spread of the jacobian ideal of an arrangement is equal to the rank of the arrangement.

Example 9. The analytic spread of the ideal $(bc + bd, ac + ad + cd + d^2, ab + bd, ab + bc + 2bd) \in \mathbb{C}[a, b, c, d]$ is 3, since it is the jacobian ideal of $(a + d)b(c + d)$

In other words, from Theorem 8 we conclude that computing the rank of a matrix (and in particular, the WLP) is equivalent to computing the analytic spread of a jacobian ideal.

Simplicial complexes and constant row sum matrices: a different perspective

Let $R = k[x_1, \dots, x_n]$ be a polynomial ring over an infinite field k (of any characteristic), set $I = (x_1^2, \dots, x_n^2) + I'$, where I' is a squarefree monomial ideal and $A = R/I$. Because I' is squarefree, the matrices that represent the multiplication by $L = x_1 + \cdots + x_n$ have a very special property:

Proposition 10. The matrix that represents $\times L : A_i \to A_{i+1}$ has constant row sum for every i.

A well known theorem in Ehrhart Theory says that the rank of such matrices is equal to the analytic spread of the monomial ideal generated by $x^{\alpha_1}, \dots, x^{\alpha_s}$ where $\alpha_1, \dots, \alpha_s$ are the rows of the matrix:

Example 11. The rank of

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

is equal to the analytic spread of the ideal

$$(x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4)$$

Moreover, using the results by Trung and Verma that connect mixed multiplicities and mixed volumes, and results from birational monomial maps by Simis and Villarreal we can show the following:

Theorem 12. Let $A = R/I$ where R is a polynomial ring in n variables and I is the sum of a squarefree monomial ideal, and the squares of the variables of R. Assume that $\dim A_1 \leq \dim A_2$. Then A either has the WLP in degree 1 in every characteristic $\neq 2$, or A fails the WLP in degree 1 in every characteristic.

Example 13. Let $I = (a^2, b^2, c^2, d^2, abd) \subset k[a, b, c, d] = R$ and set $A = R/I$. Then the matrices that must have full rank for A to satisfy the WLP are:

$$A_1 \xrightarrow{\begin{array}{c} \\ ab \\ ac \\ ad \\ bc \\ bd \\ cd \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}} A_2 \qquad A_2 \xrightarrow{\begin{array}{c} \\ abc \\ acd \\ bcd \end{array} \begin{array}{c} \begin{array}{cccccc} ab & ac & ad & bc & bd & cd \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}} A_3$$

We can then define one ideal in R_I for each matrix above:

$$I(1) = (\underbrace{t_a t_b}_{ab}, \underbrace{t_a t_c}_{ac}, \underbrace{t_a t_d}_{ad}, \underbrace{t_b t_c}_{bc}, \underbrace{t_b t_d}_{bd}, \underbrace{t_c t_d}_{cd}), \quad I(2) = (\underbrace{t_{ab} t_{ac} t_{bc}}_{abc}, \underbrace{t_{ac} t_{ad} t_{cd}}_{acd}, \underbrace{t_{bc} t_{bd} t_{cd}}_{bcd})$$

where each generator corresponds to an element in the basis of A_2 and A_3 respectively. Computing their analytic spreads we see that $\ell(I(1)) = 4$, $\ell(I(2)) = 3$ so A has the WLP in degrees $1, 2$.

Conclusions

We have seen that given a matrix, we can think of each row/column as either the exponents of a monomial, or the coefficients of a linear form. From each perspective we can associate an ideal to the matrix such that:

- The analytic spread of the jacobian ideal of the product of linear forms defined by the columns is equal to the rank of the matrix
- When the matrix has constant row sum, we can compute the rank of the matrix from the analytic spread of the monomial ideal defined by the rows as exponents of generators. Moreover, in this case, the last positive mixed multiplicity of the ideal bounds the gcd of the maximal minors of the matrix.

In particular, we have shown that determining the WLP of an artinian algebra is a question that is closely related to that of computing mixed multiplicities of jacobian ideals of products of linear forms, and equigenerated monomial ideals.

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