Lefschetz properties of squarefree monomial ideals

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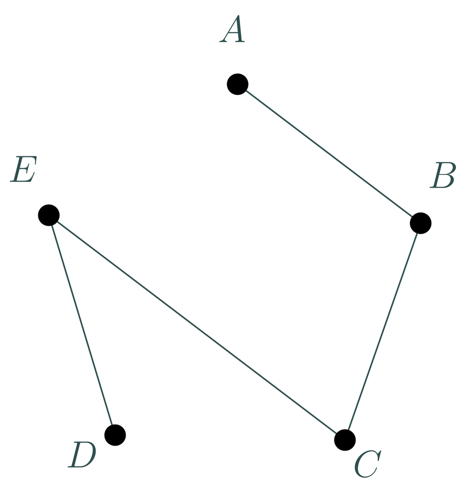
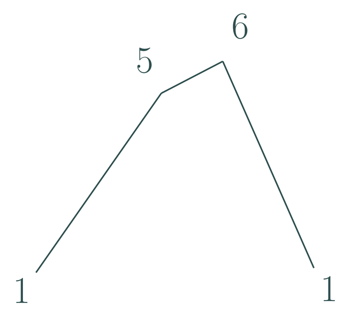
Unimodal sequences in Combinatorics

Many sequences in Combinatorics arise as solutions to the following probem:

Imagine there are 5 kids in a room, and you need to distribute n apples among them, but there are restrictions: each kid may receive at most one apple, moreover, for some pairs of kids, at most one of them can receive an apple. We can visualize the restrictions as a graph. In how many ways can you do it?



It is very common for sequences coming from combinatorics to have the property of increasing up to a point, and decreasing after. Such sequences are called **unimodal**.



Although unimodality is conjectured for many sequences, it is notoriously hard to prove that unimodality actually holds.

The Weak Lefschetz Property

The name "Lefschetz property" comes from the well known **Hard Lefschetz theorem** in geometry. At its core, the theorem says there is an isomorphism between special vector spaces associated to a geometric object. The natural question to ask, from a more algebraic perspective, is if a version of the theorem still holds even if you disconsider the geometric structure.

Definition 1. Let R be the polynomial ring over a field, and I an ideal such that $A = R/I$ is zero dimensional. We say A satisfies the **Weak Lefschetz Property** (WLP) if there exists a general homogeneous element L of degree 1, such that the multiplication maps

$$\times L : A_i \to A_{i+1} \tag{1}$$

have full rank for all i.

Surprisingly, proving that a ring A has the WLP works as a strategy to show unimodality:

$$\text{If } A \text{ has the WLP, the sequence } \dim A_0, \ldots, \dim A_d \text{ is unimodal} \tag{2}$$

A piece of a package

A stronger version of the Lefschetz properties defined above can be seen as part of a package that some artinian rings may satisfy, called the **Kähler package**, which consists of a version of the property defined above and two extra properties. The extra properties make it so the Kähler package can be used to prove **log-concavity**, which is a property stronger than unimodality that many sequences that arise in combinatorics are also conjectured to satisfy.

Recent results in this direction include for example the celebrated proof of the Rota-Heron-Welsh conjecture by Adiprasito, Huh and Katz (see [1]).

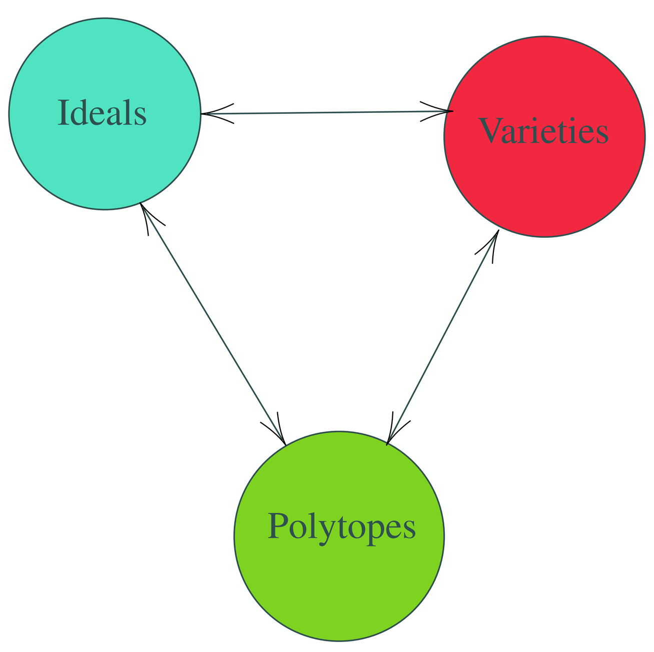
$$\text{Kähler package} \implies \textbf{log-concavity of intersection numbers} \tag{3}$$

$$\text{Lefschetz properties} \implies \textbf{unimodality of betti numbers} \tag{4}$$

Mixed multiplicities (intersection numbers, multidegrees, mixed volumes, etc)

In discrete geometry, a natural object to consider is a generalization of volume to more than one convex body, this construction is called the **mixed volume** and it has many applications in many areas of mathematics. In algebraic geometry, in order to study how curves (and more complicated objects) intersect, the notion of **intersection number** was defined. In commutative algebra, in order to better understand the geometry of intersection numbers, one can define **mixed multiplicities**.

It turns out, that all of these constructions are equivalent in many cases. In particular, many properties in discrete geometry imply very strong theorems in algebraic geometry, commutative algebra and vice-versa. Some results can also be understood from different perspectives.



The main idea is that giving a stronger structure to a simple object gives you lots of insights into how the object behaves.

	Ideals	Varieties	Polytopes
One dimension	Multiplicity	Degree	Volume
Higher dimensions	Mixed multiplicities	Multidegrees	Mixed volume

A different perspective: tying it back with linear algebra

Mixed multiplicities usually show up when trying to prove log-concavity via the Kähler package, as they form a log-concave sequence that is then shown to count something in combinatorics. It is implicit in the early results of June Huh that mixed multiplicities can be used to compute ranks of matrices.

Corollary 2. A matrix has full rank if and only if some mixed multiplicity is positive.

In particular, showing an algebra has the Weak Lefschetz Property is equivalent to showing some mixed multiplicities are positive.

In [4], we use this strategy to generalize a known result in the area, extending a characteristic 0 specific result, to positive characteristics.

Theorem 3 ([4]). Let A be an artinian algebra defined by a monomial ideal. The multiplication map from degree 1 to degree 2 by a general linear form has full rank in characteristic zero if and only if it has full rank in every odd characteristic.

Incidence matrices everywhere

In the particular case where the algebra A is defined by a squarefree monomial ideal and the squares of the variables, the multiplication maps have very particular structures.

Take for example $A = \mathbb{R}[a, b, c, d, e]/(ab, bc, ce, ed, a^2, b^2, c^2, d^2, e^2)$. Then to check if A has the WLP, one of the matrices we have to check is:

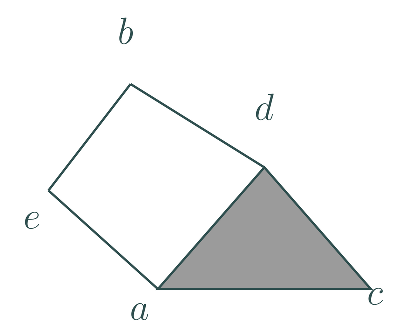
$$M = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \tag{5}$$

Now consider the rational map φ:

$$\varphi : \mathbb{P}^4 \dashrightarrow \mathbb{P}^5, \quad [a : b : c : d : e] \mapsto [ac : ad : ae : bd : be : cd] \tag{6}$$

to see if the map contains a birational map (by deleting one coordinate of $\mathbb{P}^5$) we just need to check the rank of M.

Next consider the simplicial complex Δ:



The first coboundary map of Δ is:

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix} \tag{7}$$

Notice that by removing the signs we get M.

Lastly, let G be the one skeleton of Δ, that is, the graph G where edges of G are the 1-faces of Δ. Then the signless incidence matrix of G is M.

Final remarks

Lefschetz properties are a powerful tool to study the interplay between Algebra, Geometry and Combinatorics. In particular, they play a major role in the main strategies we have of showing sequences are unimodal or log-concave.

In the particular case where the algebra A that we want to study is defined by a monomial ideal, we can identify the matrices that determine the Lefschetz properties of A with many different objects in mathematics, including:

- Boundary maps
- Rational maps
- Graphs
- Simplicial complexes
- Matroids
- Hyperplane arrangements

In doing so, we can try to use the structure of such objects to extend known results, or prove new results by exploiting the results from these other areas.

References

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