Homological invariants of ternary graphs

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Summary

In 2022, Jinha Kim proved a conjecture by Engstrom that states the independence complex of graphs with no induced cycle of length divisible by 3 is either contractible or homotopy equivalent to a sphere, which means the minimal free resolution of the edge ideal of these graphs is characteristic-free. We apply this result to give a combinatorial description of projective dimension and depth of the edge ideal of these graphs. As a consequence, we give a complete description of the multigraded betti numbers of edge ideals of ternary graphs in terms of its combinatorial structure and classify ternary graphs whose independence complex is contractible.

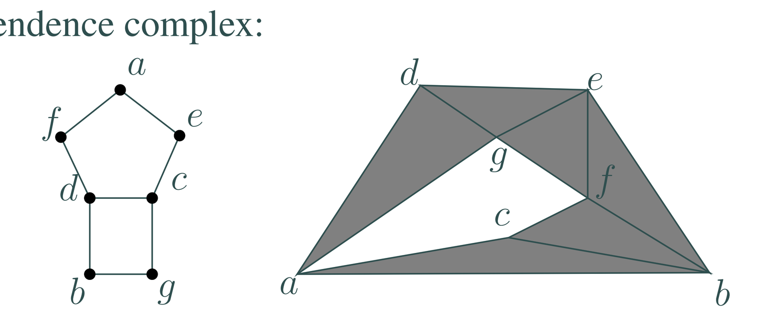
Preliminaries

Let G be a graph. A subset S of the vertices of G is said to be **independent** if there are no edges between elements of S. Since subsets of of independent sets are also independent, the independent sets of G have a simplicial complex structure.

Definition 1. Let G be a graph. The **independence complex of** G, denoted by $\text{Ind}(G)$ is the simplicial complex on vertex set $V(G)$ and the faces are the independent sets of G. Let S be a subset of the vertices of G, the induced subgraph $G[S]$ of G is the graph on vertex set S and edges $E[S] = \{uv | uv \in E(G)$ and $u, v \in S\}$.

The set $N[S] := \bigcup_{v \in S} (N(v) \cup v)$ is called the closed neighborhood of S. We write $G - S$ for the induced subgraph $G[V(G) \backslash S]$.

Example 2. A graph and its independence complex:



G $\qquad$ $\text{Ind}(G)$

Given a graph G on vertex set $\{x_1, \dots, x_n\}$ and a field k, one can define the **edge ideal** of G as follows:

$$I(G) \subset k[x_1, \dots, x_n], \; I(G) = (x_i x_j | x_i x_j \in E(G))$$

The connection between independence complexes and free resolutions of edge ideals follows from the theorem below:

Theorem 3. Let G be a graph and $x_\tau = \prod_{x_i \in \tau} x_i$ where $\tau \subset V(G)$. Then the multigraded betti numbers $b_{i,m}(R/I(G))$ of the quotient of the edge ideal of G are given by

$$b_{i,x_\tau}(R/I(G)) = \dim \tilde{H}_{|\tau|-i-1}(\text{Ind}(G[\tau]); k)$$

where $R = k[x_1, \dots, x_n]$ and $\tilde{H}$ denotes reduced simplicial homology.

Definition 4. A graph G is called **ternary** if $G[S]$ is not a cycle of length divisible by 3 for any subset S of the vertices of G.

In 2022, J. Kim proved the following theorem:

Theorem 5 (J. Kim (2022))**.** A graph G is ternary if and only if the independence complex of every induced subgraph of G is either contractible or homotopy equivalent to a sphere.

Although the theorem completely describes the independence complexes of ternary graphs, it does not say when the independence complex of a ternary graph is contractible, or the dimension of the sphere when it is not. In the particular case when the graph G is a forest, the dimension of the sphere is known by the following theorem:

Theorem 6 (M. Marietti, D. Testa (2008))**.** Let F be a forest. Then $\text{Ind}(F)$ is either contractible or homotopy equivalent to $S^{\gamma(F)-1}$ where $\gamma(F) = \min\{|S| \mid S \subset V(F), N[S] = V(F)\}$.

Using Theorem 5, we proved the following:

Theorem 7 (S. Faridi, T. Holleben (2022))**.** Let G be a ternary graph.

1. If $\text{Ind}(G)$ is not contractible, then it is homotopy equivalent to $S^{\text{depth}(\mathcal{F})-1}$, where $\mathcal{F}$ is any maximal filtration of G. Moreover, for every maximal filtration $\mathcal{F}$ the following are invariants of G:
   (a) $\text{pd}(G) := \text{del}(\mathcal{F}) + \sum_{v_i \in N(\mathcal{F})} \deg v_i$, where $\deg v_i$ is the degree of v_i as a vertex of G_i.
   (b) $\text{depth}(G) := \text{depth}(\mathcal{F})$
2. If $\text{Ind}(G)$ is contractible, we set the following notation:
   (a) $\text{pd}(G) := \max\{\text{pd}(H) | H$ is an induced subgraph with non contractible independence complex$\}$
   (b) $\text{depth}(G) := |V(G)| - \text{pd}(G)$

Leaf-filtrations and SBLs

Let G be a graph. Then for every vertex v we have the well-known **Mayer-Vietoris exact sequence for independence complexes**:

$$\cdots \to H_i(\text{Ind}(G - N[v])) \to H_i(\text{Ind}(G - v)) \to H_i(\text{Ind}(G)) \to H_{i-1}(\text{Ind}(G - N[v])) \to \cdots$$

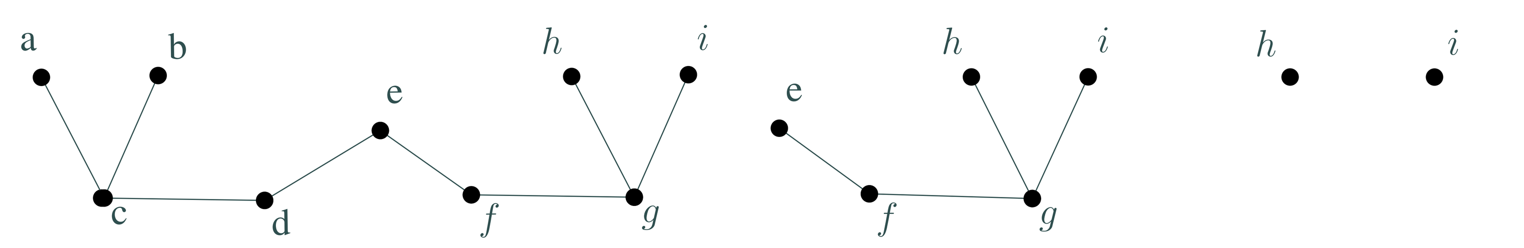
Definition 8. Let F be a forest. A **leaf-filtration** of F is a sequence of subgraphs:

$$F = F_0 \supset F_1 \supset F_2 \supset \cdots \supset F_q = \emptyset$$

where $F_i = F - N[\{v_0, \dots, v_{i-1}\}] = F_{i-1} - N[v_{i-1}]$ and v_{i-1} is adjacent to a leaf of F_{i-1} for every $i = 1, \dots, q-1$. If F_{q-1} is a graph with no edges, we define $F_q = F_{q-1} - V(F_{q-1}) = \emptyset$.

Definition 9. Let G be a graph. An edge uv of G is said to be **stuck between leaves** or SBL for short, if both u and v have degree at least 2 and are adjacent to leaves (vertices of degree one).

Example 10. The graphs below form a leaf-filtration of F, where $v_0 = c$ and $v_1 = f$.



F $\qquad$ $F - N[c]$ $\qquad$ $F - N[\{c, f\}]$

Note that the edge fg is an SBL of $F - N[c]$, but it is not an SBL of F.

Definition 11. Let F be a forest without isolated vertices. An edge uv such that uv is an SBL of a subgraph in a leaf-filtration $\mathcal{F}$ of F is called a **hidden SBL** of F.

Using the definitions above we can describe the independence complex of forests as follows.

Theorem 12. Let F be a forest. Then $\text{Ind}(F)$ is contractible if and only if F has isolated vertices or a hidden SBL. If $\text{Ind}(F)$ is not contractible, then it is homotopy equivalent to a S^{q-1} dimensional sphere, where q is the length of any leaf-filtration of F.

Moreover, if F has a hidden SBL, then any leaf-filtration of F contains a subgraph with an SBL.

Filtrations and signs of ternary graphs

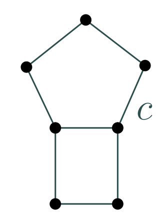
One advantage of stating Theorem 12 in the language of leaf-filtrations, is that as a consequence of Theorem 5 we can generalize the notion of leaf-filtrations to ternary graphs, and get similar results. This is because both classes of graphs (forests and ternary graphs) satisfy the following two properties:

1. Every induced subgraph of a ternary graph (resp. forest) is also a ternary graph (resp. forest).
2. Every independence complex of a ternary graph (resp. forest) is either contractible or homotopy equivalent to a sphere.

Theorem 13. Let G be a ternary graph and $S = \{v_1, \dots, v_k\}$ a set of vertices such that $G - S$ is a forest. Let $A, B \subset S$ be such that $A \cup B = S$ and $A \cap B = \emptyset$. Let $j(S)$ denote the number of forests $G - N[A] - B$ where A is an independent set and $G - N[A] - B$ does not have hidden SBLs or isolated vertices. Then the independence complex of G is contractible if and only if $j(S)$ is even.

Corollary 14. Let G be a ternary graph and S any subset of vertices such that $G - S$ is a forest. Then the number $i(G) = (-1)^{j(S)}$ is an invariant of G. We call it the **sign** of G.

Example 15. Let G be the following graph:



Then $G - c$ is a forest, and since $G - c$ does not have hidden SBLs and $G - N[c]$ has an isolated vertex, the independence complex of G is not contractible. In particular $j(\{c\}) = 1$ and thus the sign of G is -1.

Definition 16. Let G be a ternary graph with a non contractible independence complex. A filtration $\mathcal{F}$ of G is a sequence of subgraphs

$$\mathcal{F} := G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_q \tag{1}$$

with a sequence of vertices $v_0 \in V(G_0), \dots, v_{q-1} \in V(G_{q-1})$ such that

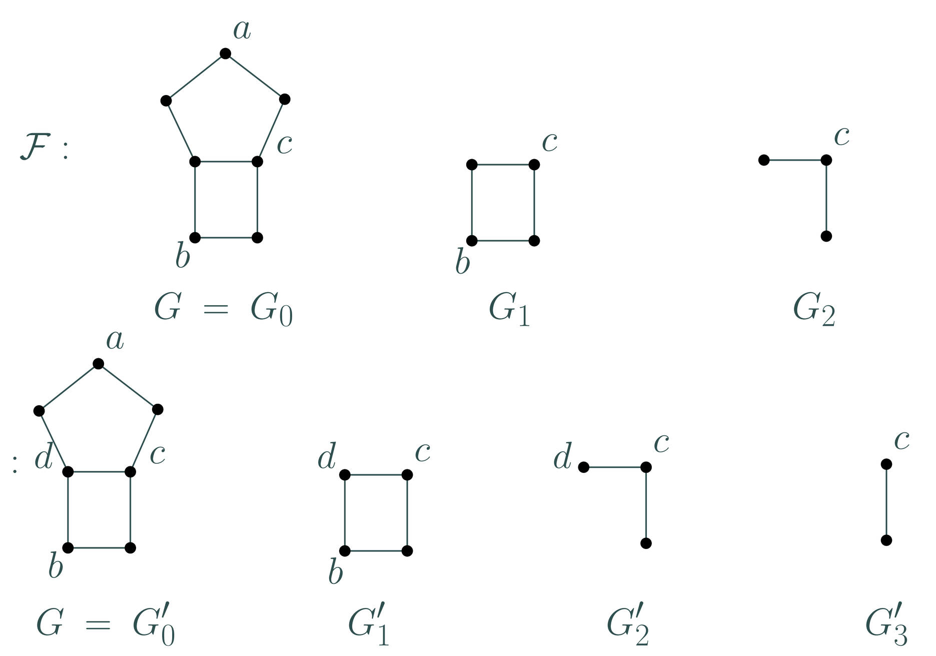
1. $G_0 = G$
2. $G_{i+1} \in \{G_i - v_i, G_i - N[v_i]\}$, where $\tilde{H}_j(\text{Ind}(G_i)) \cong \tilde{H}_{j-k_i}(\text{Ind}(G_{i+1}))$ for some $k_i \in \{0, 1\}$, $i \geq 0$.

We say a filtration is maximal if $G_q = \emptyset$.

For every filtration $\mathcal{F}$ of G we have the following definitions:

1. The **vertex deletion neighborhood** of $\mathcal{F}$ is $\text{del}(\mathcal{F}) = |\{i \mid G_{i+1} = G_i - v_i\}|$
2. The **deleted neighborhood** of $\mathcal{F}$ is $N(\mathcal{F}) = \{v_i \in G_i \mid G_{i+1} = G_i - N[v_i]\}$
3. The **depth** of $\mathcal{F}$ is $\text{depth}(\mathcal{F}) = |N(\mathcal{F})|$

Example 17. The following two sequences of subgraphs form two different maximal filtrations of the same graph G:



where

$$\mathcal{F} : G = G_0 \supset G_1 = G_0 - N[a] \supset G_2 = G_1 - b \supset \emptyset = G_2 - N[c]$$

and

$$\mathcal{F}' : G = G'_0 \supset G'_1 = G_0 - N[a] \supset G'_2 = G'_1 - b \supset G'_3 = G'_2 - d \supset \emptyset = G'_3 - N[c].$$

In particular, we have $\text{depth}(\mathcal{F}) = |\{a, c\}| = \text{depth}(\mathcal{F}')$.

Applications to Commutative Algebra

Applying Theorem 3 to the theorems above we get the following result:

Theorem 18. Let $R = k[x_1, \dots, x_n]$ and $I(G)$ the edge ideal of a ternary graph. For every squarefree monomial m let $G[m] = G[\{x_i \mid x_i | m\}]$, then

$$b_{i,m}(R/I(G)) = \begin{cases} 1 & \text{if } i(G[m]) = -1 \text{ and } i = \text{pd}(G[m]) \\ 0 & \text{otherwise} \end{cases}$$

In particular,

$$\text{pd}(G) = \text{pd}(R/I(G)) \text{ and } \text{depth}(G) = \text{depth}(R/I(G))$$

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