Lefschetz Properties of squarefree monomial ideals

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Summary

Lefschetz Properties have been studied from many different perspectives, and recently, connections to different areas of Mathematics have been shown, including Algebraic Geometry, Differential Geometry and Combinatorics. In 2021, Dao and Nair used graph theory to study the Weak Lefschetz Property (WLP) of Artinian algebras over fields of characteristic zero that are quotients of the polynomial ring by a squarefree monomial ideal, and the squares of the variables. Here we document some results about the WLP of these algebras using tools from combinatorics, and mention some connections with other areas of commutative algebra.

The Weak Lefschetz Property (WLP) and Stanley-Reisner ideals

Given a simplicial complex Δ with vertex set $[n]$, the **Stanley-Reisner** ideal of Δ is the ideal of nonfaces of Δ, that is, $I_\Delta = (\prod_{i \in \tau} x_i | \tau \notin \Delta) \subset R = k[x_1, \dots, x_n]$. From the Stanley-Reisner ideal of Δ, we also define the following artinian algebra:

$$A(\Delta) = R/(I_\Delta + (x_1^2, \dots, x_n^2)).$$

This algebra contains all the combinatorial information of Δ:

- The nonzero monomials of $A(\Delta)$ correspond to faces of Δ
- The h-vector of $A(\Delta)$ is the f-vector of Δ

We are interested in understanding the multiplication in $A(\Delta)$ by a general linear form.

Definition 1 (WLP for monomial ideals)**.** Let $L = x_1 + \dots + x_n$. We say the algebra $A := A(\Delta)$ has the **Weak Lefschetz Property**(WLP) if the multiplication maps $\times L : A_i \to A_{i+1}$ have full rank for all i.

Example 2. Let $R = k[x, y, z]$, $I = (x^2, y^2, z^2)$ and $A = R/I$. To check if $\times L : A_1 \to A_2$ has full rank, we need to check if the following matrix has full rank:

$$\begin{array}{c} \\ xy \\ xz \\ yz \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$$

One can think of the matrices that represent the maps $\times L : A(\Delta)_i \to A(\Delta)_{i+1}$ as the coboundary maps of Δ, but without the signs.

The first map

In [3], the authors applied results from graph theory to describe when does the map $\times L : A(\Delta)_1 \to A(\Delta)_2$ have full rank, assuming the base field has characteristic zero.

Theorem 3 ([3])**.** Assume $\dim A(\Delta)_1 \leq \dim A(\Delta)_2$. Then the map $\times L : A(\Delta)_1 \to A(\Delta)_2$ has full rank if and only if every connected component of the 1-skeleton of Δ has at least one odd cycle.

Assuming Δ is a d-dimensional pseudo-manifold, the authors of [3] also described when the map $\times L : A(\Delta)_d \to A(\Delta)_{d+1}$ has full rank based on the dual graph of Δ.

Theorem 4 ([3])**.** Assume the base field has characteristic 0. The map $\times L : A(\Delta)_d \to A(\Delta)_{d+1}$ has full rank if and only if one of the following conditions hold:

1. Δ has boundary
2. Δ has no boundary and the dual graph of Δ is not bipartite

Using results from birational geometry, we generalized both results to odd characteristics (see [5]).

Whiskering

One of the first examples of a simplicial complex Δ that fails the WLP is the Stanley-Reisner complex of the ideal $I_\Delta = (x_1x_2, x_1x_3, \dots, x_1x_n) \subset k[x_1, \dots, x_n]$. To see that $A(\Delta)$ fails the WLP note that:

$$(x_1 + \dots + x_n)x_1 = 0$$

In view of this example, it is natural to study how combinatorial operations on Δ that change the socle of $A(\Delta)$ affect the WLP of the respective algebras.

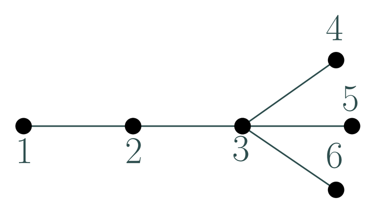
Definition 5. Let $G = (V, E)$ be a graph with $V = [n]$. The whiskering $w(G)$ is the graph with vertex set $[2n]$ and edge set $E \bigcup_{i=1}^n \{i, i+n\}$.

In joint work with S. Cooper, S. Faridi, L. Nicklasson and A. Van Tuyl we showed the following:

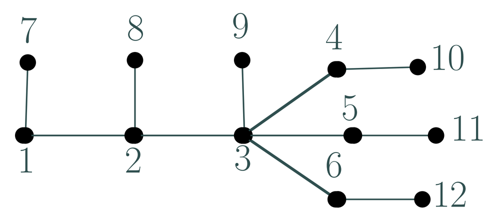
Theorem 6 ([2])**.** Let G be a graph with n vertices and at least 1 edge. Let $I \subset R$ be the edge ideal of the whiskered graph and $A = R/(I + (x_1^2, \dots, x_n^2))$. The maps $\times L : A_1 \to A_2$ and $\times L : A_{n-1} \to A_n$ have full rank whenever the base field has characteristic $\neq 2$.

Moreover, we showed that this result is optimal:

Example 7. We define the **Broom graph** $B_n = ([n+3], E)$ where $E = \{\{1, 2\}, \{2, 3\}\} \cup \{\{3, i\} | i \in 4, \dots, n+3\}$



The broom graph B_3



The whiskered broom graph $w(B_3)$

In [2], we showed the Stanley-Reisner complex of $I(w(G))$ is a pseudo-manifold with boundary, and in particular the algebra

$$A = R/(I(w(B_1)) + (x_1^2, \dots, x_8^2))$$

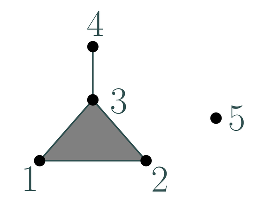
not only fails the WLP, but every map $\times L : A_i \to A_{i+1}$ does not have full rank, except for $i = 1, i = 3$.

Remark 8. Other operations that turn simplicial complexes into unmixed ones, such as grafting can also be considered. We are currently working on extending our results to grafted complexes.

Simplicial forests

A simplicial forest is a generalization of a forest (from graph theory). One consequence of Dao and Nair's result is that if Δ is a tree, then $A(\Delta)$ has the WLP. A natural question is whether the same result holds for simplicial forests. This is easily seen to be false as seen in the next example:

Example 9. Let Δ be the simplicial forest below:



Then the Stanley-Reisner ideal of Δ is $I = x_5(x_1, x_2, x_3, x_4) + x_4(x_1, x_2)$, and since $(x_1 + \dots + x_5)x_5 = 0$ in $A(\Delta)$, we see that $A(\Delta)$ fails the WLP.

The failure of the WLP in this case is due to the algebra $A(\Delta)$ having a socle element of low degree. Using incidence ideals introduced in [5] and the results from [1], we can show the following:

Theorem 10. Let Δ be a pure simplicial forest of dimension d. Then the maps $\times L : A(\Delta)_1 \to A(\Delta)_2$ and $\times L : A(\Delta)_d \to A(\Delta)_{d+1}$ have full rank whenever the characteristic is not 2.

Theorem 10 is an example of a result connecting the linear type property of an ideal I and the WLP of an artinian reduction of R/I.

Another interesting aspect of the theorem above is that it is an example of a result that gives information about the failure of the WLP in positive characteristics. In [5] we used algebraic tools (mixed multiplicities) to extend known results in characteristic zero to positive characteristics. The information about positive characteristics from Theorem 10 comes from the theory of unimodular hypergraphs.

Remark 11. The simplicial forest being pure fixes an algebraic problem: the algebra $A(\Delta)$ having a socle element of low degree. In the previous section we mention a combinatorial operation (whiskering) that fixes this problem and some of its consequences to the WLP.

Another approach is to study algebraic operations that fix this problem. In [8], the authors defined the notion of *levelable* algebras, by finding a different monomial artinian reduction of I_Δ that gives a level algebra. Natural questions then include the connection between leveling an algebra and Lefschetz properties, and unimodality of the respective h-vectors.

For example, we can specify a different artinian reduction for the two examples of failures of the WLP given so far such that they satisfy the WLP, and the unimodality of the h-vector of the new algebra implies the unimodality of the h-vector of $A(\Delta)$:

$$A' = R/((x_1^2, x_2^2, x_3^2, x_4^2, x_5^4) + I_\Delta)$$

Squarefree Gotzmann ideals

A homogeneous ideal of R generated in degree d is **Gotzmann** if its Hilbert function is minimal among all homogeneous ideals generated in the same degree and same number of elements.

The WLP of Gotzmann ideals was first studied in [9], where the author gave necessary and sufficient conditions for $\mathfrak{m}$-primary Gotzmann ideals to have the Weak Lefschetz property based on their Hilbert functions.

In [4], the authors completely described squarefree Gotzmann ideals as ideals of the form:

$$I = m_1(x_{i_{1,1}}, \dots, x_{i_{s_1,1}}) + m_1m_2(x_{i_{1,2}}, \dots, x_{i_{s_2,2}}) + \dots + m_1 \dots m_r(x_{i_{1,r}}, \dots, x_{i_{s_r,r}})$$

all having pairwise disjoint supports.

With this description of the generators of a squarefree Gotzmann ideals, we can ask whether the squarefree reduction of a Gotzmann ideal has the WLP.

These ideals can be decomposed as either $I = x_iI'$ or $I = x_i + I'$, where I' is a squarefree Gotzmann ideal of $k[x_1, \dots, \hat{x}_i, \dots, x_n]$. Since every homogeneous ideal of $k[x, y]$ has the WLP, we can inductively make many examples of Gotzmann ideals satisfying the WLP.

Example 12. Take for instance the ideal

$$I = x_1x_5(x_2) + x_1x_5x_3(x_4) \subset R = \mathbb{Q}[x_1, \dots, x_5].$$

Then the algebra $A = R/(I + (x_1^2, \dots, x_5^2))$ satisfies the WLP.

The "star shape" of the generators can also be exploited to generate examples that fail the WLP due to the quotient having a socle element of low degree.

Example 13. Let $I = x_1(x_3, \dots, x_9) + x_1x_{10}(x_{11}) \subset R = \mathbb{Q}[x_1, \dots, x_{11}]$. Then

$$(x_1 + \dots + x_{11})x_1x_{10} = 0 \quad \text{in } A = R/(I + (x_1^2, \dots, x_{11}^2))$$

and since $\dim A_2 \leq \dim A_3$, we see that A fails the WLP.

Remark 14. Note that the reason for failure of the WLP was again a socle element of low degree. It is again natural to ask whether this can be fixed by either considering combinatorial operations (whiskering, grafting, etc) that give pure simplicial complexes, or algebraic operations that give level algebras such as leveling.

References

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