THE FAILURE OF LEFSCHETZ PROPERTIES: A COMBINATORIAL PERSPECTIVE

by

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Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy

at

Dalhousie University
Halifax, Nova Scotia
June 2026

"I hope the readers of this [thesis] enjoy reading it as much as I enjoyed writing it. For me, this project was a giant puzzle—an academic scavenger hunt. Finding the pieces and assembling them into a cohesive story was a challenge and a joy for me. I love my job."

DAVE RICHESON – *"Euler's gem"*

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Abstract

How does one prove that a given sequence of numbers is increasing up to some point, and then decreasing? In 1980, Stanley used a well-known theorem from algebraic geometry in order to show that the number of faces in a simplicial polytope has this property. This result led to the study of what are now called "Lefschetz properties" in commutative algebra. In this thesis, we explore questions related to the study of Lefschetz properties from a combinatorial perspective. Inspired by geometric results of Cook-Harbourne-Migliore-Nagel, we explore the failure of Lefschetz properties of certain monomial algebras, and show that an algebra failing Lefschetz properties can be used to construct systems of parameters. We apply this idea in order to show that certain monomial algebras fail Lefschetz properties with high probability. We also explore results from graph theory in order to prove the existence of extreme examples of Hilbert series of Gorenstein artinian algebras. Finally, we consider Betti numbers of squarefree monomial ideals, which are finer invariants than the Hilbert series. We introduce the notion of a "spherical complex", and provide a method for computing minimal free resolutions of their Stanley-Reisner ideals.

Acknowledgements

First of all, I would like to thank my advisor, Sara Faridi. In these past four years, she was always there for me and ready to help with whatever I needed, from very long meetings to tips on how to book cheap flights. I am very grateful for all her support. She taught me what it means to be a mathematician beyond just proving theorems, and I am very lucky to have been her student.

I would also like to thank my master's advisor, Hamid Hassanzadeh. I was first introduced to many areas of mathematics in his algebra classes. His classes and the time I spent as his student deeply influenced the way I think about mathematics.

Throughout my PhD, I had the opportunity to meet and collaborate with many amazing people. I would like to thank Nasrin Altafi, Yirong Yang, and Yufei Zhang for introducing me to several topics that are central to this thesis.

Towards the end of my PhD, I visited Professor Martina Juhnke twice. The time I spent at Osnabrück University was extremely productive and fun. I would like to thank Martina Juhnke, Torben Donzelmann, and the entire polytopes group for the many enlightening discussions.

I would also like to thank my friends for all their support. E, por último, quero agradecer à minha família por todo o apoio e por sempre acreditar em mim.

Chapter 1

Introduction

"When Coleridge tried to define beauty, he returned always to one deep thought: beauty, he said, is "unity in variety." Science is nothing else than the search to discover unity in the wild variety of nature-or more exactly, in the variety of our experience. [...] What is a poetic image but the seizing and the exploration of a hidden likeness, in holding together two parts of a comparison which are to give depth each to the other?"

JACOB BRONOWSKI – *Science and Human values*

Combinatorics and topology are two areas of mathematics that have been interlaced at least since the late 1890s, due to Poincaré's work on manifolds. On the one hand, one of the first applications of topological methods to solve a question in combinatorics goes back to 1978, where Lovász used tools from algebraic topology in order to prove Kneser's conjecture, about the chromatic number of a specific class of graphs. Going in the other direction, applications of combinatorial tools to topology can be traced back at least to the early stages of algebraic topology at the end of the nineteenth century, where the main focus was to understand invariants of manifolds (e.g. surfaces), and the strategy was to essentially show that some number associated to a manifold was independent of the triangulation, and then work with a simple triangulation. A famous example of this phenomenon can be seen in the Gauss-Bonnet theorem, where the total curvature of a compact orientable surface M is equal to $2\pi\chi(M)$, where $\chi(M)$ is the Euler characteristic of M, and can be understood purely combinatorially from any triangulation of M.

In this thesis, we consider three types of questions: the ones arising from combinatorics where topological tools turn out to be useful, ones arising from topology where combinatorics turns out to be useful, as well as algebraic questions that follow from understanding the previous two types of questions in an algebraic setting.

1.1 Commutative algebra's role in the intersection of topology and combinatorics

1.1.1 The beginning: face numbers and ideals of nonfaces

Given a simplicial complex Δ, the *face numbers* f_i of Δ count the number of i-dimensional faces of Δ. The sequence $(f_0, \ldots, f_d)$ is called the f*-vector* of Δ. Even though face numbers are the most natural way to consider the enumerative information in a simplicial complex, it is often the case that relevant properties are easier to state in a more condensed version of the f-vector called the h*-vector*, which is obtained from the f-vector by certain polynomial relations. The polynomials with coefficients arising from the f- and h-vectors are called the f*-polynomial* and h*-polynomial*, respectively.

In 1975 Stanley [181] showed the h-polynomial of a simplicial complex Δ is the numerator of the *Hilbert series* of an algebra A, which is the numerator of the rational function that is the generating function of the graded pieces of A. This algebra is nowadays called the *face ring* or *Stanley-Reisner ring* of Δ, and its defining ideal is called the *Stanley-Reisner ideal* of Δ. In view of this observation, several tools from commutative algebra became available for the study of inequalities of f- and h-vectors of Δ.

The earliest application of the idea above can be found already in [181], where Stanley used a criterion due to Reisner [171] that characterizes Cohen-Macaulay face rings and showed the following.

Theorem 1.1.1 (Upper Bound Theorem, [181]**).** *Let Δ be a $(d-1)$–dimensional simplicial complex with n vertices and h-vector $(h_0, \ldots, h_d)$. Then if the face ring of Δ is Cohen-Macaulay, Δ satisfies the upper bound conjecture:*

$$h_i \leq \binom{n-d+i-1}{i} \quad \textit{for} \quad 0 \leq i \leq d.$$

In particular, the result holds when Δ is a triangulation of a ball or sphere.

After Stanley's proof of the Upper bound conjecture, it quickly became clear that commutative algebra was a powerful tool in order to understand, prove and even state questions arising in f-vector theory. One of the main reasons the two topics are so intertwined is due to a classification result due to Macaulay in 1927 [127] (see also Section 2.4 and [27, Theorem 4.2.10]), which classifies the sequences of numbers that can arise as the Hilbert series of a standard graded algebra. In view of Macaulay's result, many statements and questions in f-vector theory can be expressed using the language of Hilbert series. Perhaps the most famous such example is the g-conjecture: a list of 3 properties proposed by McMullen in 1971 [137] that conjecturally classifies the possible h-vectors of homology spheres.

Conjecture 1.1.2 (**McMullen's g-conjecture**[1]). *A sequence $(h_0, \ldots, h_d)$ is the h-vector of a $(d-1)$–dimensional homology sphere if and only if*

1. $h_0 = 1$
2. $h_k = h_{d-k}$ *for* $0 \leq k \leq d$ *(Dehn-Sommerville equations* [75, Section 9.2]*)*
3. *The sequence given by $g_0 = 1$ and $g_i = h_i - h_{i-1}$ for $0 < i \leq \lfloor d/2 \rfloor$ is the Hilbert series of a standard graded algebra.*

Remark 1.1.3. The Dehn-Sommerville equations were already known since the 1920s. In dimension 2, the case $k = 2$ in 2. is a restatement (using h_i instead of f_i) of a famous result due to Euler that says $v - e + f - 1 = 1$ for a polytope in $\mathbb{R}^3$ with v vertices, e edges and f faces.

The original statement of 3. used the notion of "pseudopowers" that was also used by Macaulay 44 years earlier, in order to characterize sequences arising as Hilbert series of standard graded algebras[2]. The fact that condition 3. is surprising was acknowledged by McMullen and Shepard [138, pp. 178] when they write the following right before stating the g-conjecture: "Even more intriguing, if rather less plausible, is the conjecture proposed in [137]".

Since 1971, a lot of the research around f-vectors of homology spheres has been focused on trying to prove McMullen's g-conjecture. In 1981 Billera and Lee [13] showed the sufficiency part of the conjecture by constructing boundary complexes of simplicial polytopes for each sequence $(h_0, \ldots, h_d)$ satisfying the assumptions above, and in 1980 Stanley [185] showed the necessity part of the conjecture for simplicial polytopes. Stanley's result became known as the *g-theorem*. It was not until 2021 that Adiprasito, Papadakis and Petrotou announced a complete proof of the conjecture (see also [3, 4, 108, 158]). We

[1] The g-conjecture was originally stated only for the class of boundary complexes of simplicial polytopes. The general version of the statement can be found in Stanley's paper [182, pp. 58]

[2] It is sometimes mentioned that McMullen was unaware of Macaulay's results from 1927 relating the original statement of 3. to algebra. The original statement of 3. defined the g_i in terms of f-vectors instead, and stated the equivalent inequalities from Macaulay's theorem.

note however, that between Stanley's proof of the g-theorem and 2021, there were several related results, such as Murai and Nevo's proof of the generalized lower bound theorem [153], and Juhnke-Kubitzke and Murai's proof of the balanced generalized lower bound theorem [101].

The main strategy in most results around the g-conjecture stems from Stanley's original proof of the g-theorem, which used a version of the following famous theorem in algebraic geometry.

Theorem 1.1.4 (**Hard Lefschetz Theorem**). *Let X be a smooth complex irreducible projective variety and $H^*(X) = H^0(X) \oplus \cdots \oplus H^{2n}(X)$ its singular cohomology ring. Then there exists an element $\omega \in H^2(X)$ such that the multiplication maps*

1. $\times\omega : H^i(X) \to H^{i+2}(X)$ *is an injective linear transformation for* $i \leq n-1$
2. $\times\omega : H^i(X) \to H^{i+2}(X)$ *is a surjective linear transformation for* $i \geq n$
3. $\times\omega^{n-i} : H^i(X) \to H^{2n-i}(X)$ *is a bijective linear transformation for* $i \leq n$.

Stanley's original insight was to use the known fact that there exists a special sequence of linear forms $\theta_1, \ldots, \theta_d$ associated to the boundary complex of a simplicial polytope Δ, such that the cohomology ring $A = H^*(X)$ (an artinian algebra) of a certain toric variety X is presented exactly by the quotient of the Stanley-Reisner ring of Δ by the linear forms $\theta_1, \ldots, \theta_d$. The set of linear forms $\theta_1, \ldots, \theta_d$ is a special example of a *system of parameters*, a crucial concept in the dimension theory of commutative rings.

A key obstruction to generalizing Stanley's proof of the g-theorem is that there is no toric variety associated to an arbitrary homology sphere, and in particular, there is no special system of parameters as above. A common theme in the recent results around the g-conjecture for homology spheres is to consider a quotient of the Stanley-Reisner ring of a homology sphere by *general linear forms*, which are known to form a linear system of parameters.

The g-theorem was one of the first results in an area of commutative algebra that is now called the study of *Lefschetz properties* of artinian algebras. An artinian algebra A is said to satisfy the *strong Lefschetz property* if there exists a linear form $\ell \in A_1$ such that multiplication maps $\times\ell^j : A_i \to A_{i+j}$ have full rank for every i, j.

One of the main focuses of this thesis is to study the following broad question, which is motivated by McMullen's g-conjecture.

Question 1.1.5. *Let the simplicial complex Δ be a homology sphere and $I_\Delta \subset R$ its Stanley-Reisner ideal. How can we generate sequences of (not necessarily linear) homogeneous elements $\theta_1, \ldots, \theta_d \in R$ that form a system of parameters of I_Δ?*

Our main approach to study Question 1.1.5 can be understood via an analogy to the recently developed theory of unexpected hypersurfaces, introduced by Cook, Harbourne, Migliore and Nagel in 2018 [32]. We introduce the notion of an *unexpected system of parameters* of a Cohen-Macaulay complex Δ, and show that special classes of rings that have been studied in combinatorics coincide with quotients of Stanley-Reisner rings of homology spheres by unexpected systems of parameters. These connections lead us to the following question, which generalizes a conjecture by Cook, Juhnke, Murai and Nevo [34] on the Lefschetz properties of a class of spheres called *balanced spheres* when the underlying system of parameters is the *colored system of parameters*.

Question 1.1.6. *Let Δ be a homology sphere, $I_\Delta \subset R$ its Stanley-Reisner ideal and $\theta_1, \ldots, \theta_d \in R$ an unexpected system of parameters of Δ. Does the ring $\frac{R}{I_\Delta + (\theta_1, \ldots, \theta_d)}$ satisfy the strong Lefschetz property?*

1.1.2 A different beginning: ideals of faces, flag complexes and Koszulity

In 1990 Villarreal [197] introduced a different method of associating an ideal to a combinatorial object. Villarreal's construction, which was later generalized by Faridi [53] in 2002, is nowadays called the *edge* or *facet* ideal of a graph or simplicial complex[3] is in many ways a more natural construction: it associates monomials to each maximal face of a simplicial complex, instead of minimal nonfaces.

The key difference between the two theories is that while Stanley-Reisner theory focuses on gathering all of the "irrelevant" combinatorial information (nonfaces) in an ideal, so that only the relevant information that survives in the quotient, facet ideal theory focuses all of the relevant information in the ideal itself. These two perspectives can be observed very clearly in the literature: Stanley-Reisner theory is often associated with algebraic questions arising from quotient rings (e.g. Hilbert series), while facet ideal theory is often studied in the context of properties of ideals (e.g. properties of powers of ideals).

Even though both theories are often applied to different questions in commutative algebra, they are very much related from a combinatorial perspective. One of the most

[3]Sometimes in the literature the language of hypergraphs is used instead of simplicial complexes

well-studied classes of simplicial complexes are the so called *flag complexes*, which are complexes that arise from independent sets of graphs. In a more algebraic language, flag complexes are the complexes whose Stanley-Reisner ideal is generated in degree 2. Equivalently, flag complexes are the Stanley-Reisner complexes of edge ideals of graphs.

From a graph theoretical perspective, the enumeration of faces of flag complexes is equivalent to the study of independence polynomials of graphs, and has a rich history. In 1987 Alavi, Malde, Schwenk and Erdős [5] showed that the coefficients of independence polynomials of graphs satisfy no natural inequality that holds for all graphs. Later in 2000, Brown, Dilcher and Nowakowski [24] conjectured that for the class of "well-covered graphs"[4] the coefficients of the independence polynomial were unimodal, that is, the coefficients $f_0, \ldots, f_d$ satisfy the inequalities

$$f_0 \leq \cdots \leq f_{k-1} \leq f_k \geq f_{k+1} \geq \cdots \geq f_d \quad \text{for some} \quad k.$$

This conjecture was shortly disproven by Michael and Traves in 2003 [143], where they show the first half of coefficients is increasing, and conjectured the second half is "unconstrained". Their conjecture became known as the *Roller Coaster conjecture*, and was proven in 2017 by Cutler and Pebody [37].

Theorem 1.1.7 (**Roller Coaster theorem for well-covered graphs**, [37])**.** *Let $q > 0$ be an integer and π a permutation of $\{\lceil q/2 \rceil, \ldots, q\}$. Then there exists a well-covered graph G with independence polynomial $f_0 + f_1 t + \cdots + f_q t^q$ such that*

1. $f_0 < \cdots < f_{\lceil q/2 \rceil}$ *(increasing up to the middle,* [143]*)*

2. $f_{\pi(\lceil q/2 \rceil)} < f_{\pi(\lceil q/2 \rceil + 1)} < \cdots < f_{\pi(q)}$ *(unconstrained after the middle,* [37]*)*

From a more topological perspective, in 1987 Gromov [74] used the notion of flag complexes in order to characterize cubical complexes satisfying a certain condition on their curvature. This class is known as the class of CAT(0) cubical complexes. In 1995, Charney and Davis [28] used Gromov's characterization in order to state a discrete version of a famous conjecture due to Hopf on the sign of the Euler characteristic of special Riemannian manifolds.

[4]Flag complexes arising from well-covered graphs are also called *pure* (flag) simplicial complexes

Conjecture 1.1.8 (Charney-Davis conjecture, [28, Conjecture D]**).** *Suppose that L is a simplicial complex homeomorphic to an $(2n+1)$-dimensional sphere. If L is a flag complex, then*

$$(-1)^n \kappa(L) \geq 0 \quad \textit{where} \quad \kappa(L) = 1 + \sum_i \left(\frac{-1}{2}\right)^{i+1} f_i$$

and the f_i are the entries of the f-vector of L (or equivalently, the coefficients of the independence polynomial of a graph).

The Charney-Davis conjecture (if true) is a special inequality that flag spheres must satisfy, but it is not satisfied by arbitrary spheres. In 2004, Davis et. al. [41] conjectured that the h-polynomial of a flag generalized homology sphere (for example, a flag complex homeomorphic to a sphere) only has real roots. If true, their conjecture would imply the Charney-Davis conjecture. Shortly after however, in 2005, Gal [64] showed that the h-polynomial of a flag generalized homology sphere (GHS for short) can have nonreal roots, and conjectured that instead, the h-polynomial of a flag GHS satisfies a weaker property called γ-*positivity*.

Conjecture 1.1.9 (Gal's conjecture, [64, Conjecture 2.1.7]**).** *Let Δ be a flag GHS with h-polynomial $h_\Delta(t)$. Then the polynomial γ_Δ such that*

$$h_\Delta(t) = (1+t)^n \gamma_\Delta\Big(\frac{t}{(1+t)^2}\Big)$$

has nonnegative coefficients.

Gal's conjecture should be seen as the flag analogue of the *generalized lower bound inequalities* for polytopes and spheres, proven by Murai and Nevo in 2013 [153], which state (in particular) that the g_i from Conjecture 1.1.2 are nonnegative.

Finally, from a more algebraic perspective, flag complexes are intrinsically related to the theory of *Koszul algebras*. A $\mathbb{K}$-algebra A is said to be Koszul if the minimal free resolution of the underlying field $\mathbb{K}$ is linear over A.

When A is the quotient of a polynomial ring over $\mathbb{K}$ by a monomial ideal I, A is Koszul if and only if I is generated in degree 2 (see [62] for more general statements). In particular, Koszul monomial algebras are exactly the algebras associated to Stanley-Reisner ideals of flag complexes[5]. In view of this connection, studying h-vectors of flag complexes

[5]Equivalently, algebras associated to edge ideals of graphs

is equivalent to studying Hilbert series of certain Koszul algebras.

The Hilbert series of Koszul algebra have been extensively studied, and it is known that these sequences satisfy special inequalities [164, Chapter 7]. In a survey paper from 2009, Peeva and Stillman [160, Problem 10.3] asked whether an "algebraic" version of Conjecture 1.1.8 holds. More concretely:

Question 1.1.10 (**Algebraic Charney-Davis conjecture,** [160, Problem 10.3]). *If A is an artinian Gorenstein Koszul algebra with h-vector $(h_0, \dots, h_{2e})$, is it true that $(-1)^e(h_0 - h_1 + h_2 - \cdots + h_{2e}) \geq 0$?*

A negative answer to Question 1.1.10 was given in 2021 by D'Alì and Venturello [38], when the authors constructed artinian Gorenstein Koszul algebras arising from (surprisingly) Cohen-Macaulay flag simplicial complexes that serve as counterexamples to Question 1.1.10. The construction from [38] combines two known constructions, one from combinatorics, and one from algebra:

1. The Bier ball associated to a simplicial complex, and
2. the Nagata idealization of a level artinian algebra

We note that it is not surprising that Nagata idealization is used in order to construct badly behaved algebras. The same construction was used by Stanley in 1978 [183] in order to find the first example of a nonunimodal h-vector of an artinian Gorenstein algebra, and later by Boij [17] in order to construct h-vectors of artinian Gorenstein algebras with arbitrarily many valleys.

In this direction, in joint work with Lisa Nicklasson, we consider Bier's construction from a graph theoretic perspective (in this language also known as whiskering, leafy extension or corona product), as well as Nagata idealization, in order to prove the following result:

Theorem 1.1.11 (**Roller Coaster theorem for artinian Gorenstein algebras**, [96]). *Given a positive integer d and a permutation π of the numbers $\{1, \dots, \lfloor d/2 \rfloor\}$, there exists an artinian Gorenstein algebra with h-vector $(h_0, \dots, h_d)$ such that*

$$h_{\pi(1)} < \cdots < h_{\pi(\lfloor d/2 \rfloor)}.$$

In other words, the first half of Hilbert series of an artinian Gorenstein algebra is unconstrained.

1.1.3 Leray numbers and Castelnuovo-Mumford regularity, graph cycles and spheres

In 1913[6] Helly [84] proved the following classical theorem in convex geometry:

Theorem 1.1.12 (**Helly's theorem** [84])**.** *If each $d+1$ members of a finite family of at least $d+1$ convex sets in $\mathbb{R}^d$ have nonempty intersection, then the whole family has a nonempty intersection.*

In a language more similar to the one we have used so far, we may associate to every collection of convex sets $\mathcal{F}$ in $\mathbb{R}^d$ a simplicial complex $N(\mathcal{F})$ called the *Nerve of* $\mathcal{F}$, where the vertices of $N(\mathcal{F})$ correspond to each convex set in $\mathcal{F}$, and a set σ is a face of $N(\mathcal{F})$ if the convex sets in σ have a nonempty intersection. A simplicial complex that can arise under this construction is called a d*-representable complex*. Helly's theorem should be thought of as a necessary condition that d-representable complexes must satisfy: if the boundary of a k-simplex is a face of Δ for some $k \geq d$, then the $(k+1)$–simplex itself must also be a face of Δ. Note that for $d = 1$ the condition is equivalent to $N(\mathcal{F})$ being a flag complex. For $d = 1$, a complete characterization of d-representable complexes was obtained by Lekkerkerker and Boland in 1962 [117], this class is now known as (the clique complexes of) *interval graphs*.

A related (more topological) result due to Leray in 1945 [118] gives another necessary condition for a complex Δ to be d-representable:

Theorem 1.1.13 (**Nerve theorem for homology groups** [118])**.** *If a space X is covered by a finite family $\mathcal{F}$ of cells such that every nonempty intersection of collections of elements in $\mathcal{F}$ is also a cell, then $N(\mathcal{F})$ and X have isomorphic homology groups.*

In particular, every induced subcomplex of a d-representable complex has trivial j-homology groups for every $j \geq d$.

With Leray's Nerve theorem in mind, in 1975 Wegner [201] introduced the class of d-*Leray complexes* exactly as the class of complexes Δ such that every induced subcomplex

[6] Although the paper was only published later, an explanation for the date 1913 instead of 1923 or 1930 can be found in [39]

of Δ has trivial j-homology for $j \geq d$. Following Wegner's idea, the Leray number of Δ is the smallest number $L_{\mathbb{K}}(\Delta)$ such that Δ is $L_{\mathbb{K}}(\Delta)$-Leray when homology is taken with coefficients over the field $\mathbb{K}$.

Computing the Leray number of a simplicial complex is often a hard task. In commutative algebra, in view of a classical result due to Hochster [89] from 1975, computing the Leray number of Δ over a field $\mathbb{K}$ turns out to be equivalent to computing an invariant called the *Castelnuovo-Mumford regularity* (or simply, the regularity) $\operatorname{reg}(R/I_\Delta)$ of the Stanley-Reisner ring of Δ, where $\mathbb{K}$ is the base field of the underlying polynomial ring. Similarly, in a more algebraic language Helly's theorem is the statement that if Δ is d-representable, then I_Δ only has generators of degree $\leq d$.

In view of the natural and clean equivalence between the topological notion of Leray numbers and the algebraic notion of regularity, there are several topological results about Leray numbers that can be interpreted in more algebraic terms and vice-versa. One of the many examples of this phenomenon is a conjecture from 2001 due to Terai [194, Conjecture 2.2] on the regularity of sums of squarefree monomial ideals, that was proven in 2006 by Kalai and Meshulam [106] using Leray numbers.

From a more graph theoretical perspective, homological conditions on complexes arising from a graph G have often been viewed as obstructions to a graph having a high (or low) chromatic number. This type of idea was first used by Lovász [123, Theorem 2], who showed that if a certain topological space associated to a graph G is not $(k+2)$–connected, then G does not admit a proper coloring with k colors. Since the 1978 paper by Lovász, there have been several results which follow the same philosophy of finding topological obstructions to graph coloring. We refer the reader to [190, Figure 1] and [134] for more details on this topic.

With the interpretation of topological invariants as chromatic obstructions in mind, Kalai and Meshulam [105] conjectured the following:

Conjecture 1.1.14 (Kalai-Meshulam homological conjectures). *Given a graph G, denote by $\operatorname{Ind}(G)$ the independence complex of G and by $\beta(\operatorname{Ind}(G))$ the sum of the reduced Betti numbers of $\operatorname{Ind}(G)$.*

1. *For every $c > 0$ there exists a constant $K(c)$ such that for every graph G, if $\beta(\operatorname{Ind}(H)) \leq c$ for every induced subgraph H of G, then G can be colored by $K(c)$ colors.*

2. *For a graph G, we have $\beta(\mathrm{Ind}(H)) \leq 1$ for every induced subgraph H of G, if and only if G does not contain induced cycles of length divisible by* 3.

A graph G is said to be *ternary* if G does not have induced cycles of length divisible by 3, so that the second part of the conjecture says G is ternary if and only if $\beta(\mathrm{Ind}(H)) \leq 1$ for every induced subgraph of G.

The second part of the Kalai-Meshulam conjecture can be seen as the special case of the first part of the conjecture where $c = 1$ and in this case the class of graphs satisfying the property is also classified. In this direction, Bonamy, Charbit and Thomassé [18] showed in 2014 that if a graph is ternary, then it must have bounded chromatic number[7], providing strong evidence for the conjecture in the case $c = 1$.

In terms of the topological aspect of the conjecture, in 2020 Engström [49] showed that the independence complex of a graph without cycles of length divisible by 3 is either contractible or homotopy equivalent to a sphere. As a consequence, $\beta(\mathrm{Ind}(H)) \leq 1$ for every induced subgraph H of a ternary graph G. The converse was shown shortly after by Zhang and Wu [207], who showed $\beta(\mathrm{Ind}(H)) \leq 1$ for every induced subgraph H of G holds if and only if G is ternary. In 2022 Kim [109] then showed that the independence complex of every induced subgraph of G is either contractible or homotopy equivalent to a sphere if and only if G is ternary.

Inspired by Kim's (topological) classification of ternary graphs, in joint work with Sara Faridi [57], we introduce the class of *spherical complexes*: a simplicial complex Δ is spherical if every subcomplex of Δ obtained by taking links and deletions repeatedly is either acyclic or has the homology of a sphere. This class of complexes also includes for example simplicial trees, introduced by Faridi [53] in 2002, in the same paper where she defines facet ideals.

The condition used to define spherical complexes should be seen as a generalization of the notion of a homology sphere. A *homology sphere* is a d-dimensional simplicial complex Δ such that Δ has the same homology as a d-sphere, and every link of Δ has the homology of a sphere of the highest possible dimension.

For spherical complexes, on the one hand we strengthen the condition that every link has the homology of a sphere, by also imposing that any complex obtained via links and

[7] In [18] the authors write: "We did not try to figure out the bound we can derive from our method, but we are pretty confident that its decimal expansion should easily fit in one line. However, we feel a little bit sorry since the threshold seems to be 4"

deletions also has the homology of a sphere. On the other hand, we loosen the definition by allowing such complexes to have the homology of a sphere of arbitrary dimension.

For homology spheres, a foundational result of Stanley [187] says that homology spheres correspond exactly to Gorenstein Stanley-Reisner rings arising from nonacyclic complexes. For nonacyclic spherical complexes, we show the following:

Theorem 1.1.15 ([57])**.** *Let Δ be a nonacyclic spherical complex over a field $\mathbb{K}$. If Δ has the homology of a d-dimensional sphere over $\mathbb{K}$, then the Leray number of Δ over $\mathbb{K}$ is equal to $d+1$.*

In algebraic terms, the Stanley-Reisner ideal of Δ satisfies:

$$\operatorname{depth}(R/I_\Delta) = \operatorname{reg}(R/I_\Delta),$$

where $\operatorname{depth}(R/I_\Delta)$ is the size of the largest regular sequence of I_Δ.

At the end of [57], we asked the following question, which tries to classify homology spheres that are spherical.

Question 1.1.16. *Let Δ be a d-dimensional spherical complex. Are the following equivalent?*

1. *Δ has the homology of a d-dimensional sphere, and*
2. *Δ is a homology sphere.*

In recent joint work with Margaret Bayer, Richard Danner, Marie Kramer and Yirong Yang [10] we give a positive answer to this question, and show that the independence complex of every planar ternary Gorenstein graph is in fact the boundary complex of a vertex decomposable simplicial polytope.

Chapter 2

Preliminaries

> *"[...] but in the transition from romantic to actual I have learned many things I should never have known had I not tried the experiment. One of them is the precious science of patience, which teaches us that we should take our education as we would take a walk in the country, leisurely, our minds hospitably open to impressions of every sort. Such knowledge floods the soul unseen with a soundless tidal wave of deepening thought."*
>
> HELEN KELLER – *The story of my life*

This chapter is meant to include some of the (broad) background required for the rest of this text. For more details and proofs of the results in this section we refer the reader to: [80, 133] (for Section 2.1), [133, 187] (for Section 2.2), [14, 99, 133, 159] (for Section 2.3) and [27, 78, 135, 187] (for Sections 2.4 and 2.5). The reader should feel free to skip sections where the content is familiar. More specific background will be included in the corresponding sections.

2.1 Topological background

In this section we introduce the topological background, including the concept of *simplicial homology*. The notion of homology has a beautiful history and it is (somewhat surprisingly) motivated by problems arising in the study of contour integration. We refer the reader to [202] (see also [173, Chapter 1]) for more historical details.

Definition 2.1.1 (Simplicial complexes: basic notions and terminology). A *simplicial complex* Δ on vertex set V is a finite collection of subsets of V such that if $\tau \in \Delta$ and $\sigma \subset \tau$, then $\sigma \in \Delta$. Subsets $\tau \in \Delta$ are called *faces*. The *dimension* of a face is $\dim \tau = |\tau| - 1$. Zero dimensional faces of Δ are called vertices, 1-dimensional faces are called edges and

2-dimensional faces are called triangles. The dimension of Δ is $\dim\Delta = \sup(\dim\tau\colon \tau \in \Delta)$. A one dimensional simplicial complex G is also called a *graph* and maybe written as a tuple $G = (V, E)$ of vertices and edges. The simplicial complex

$$\Delta^{[k]} = \{\sigma \in \Delta\colon |\sigma| \leq k+1\}$$

of all faces of Δ of dimension at most k is called the *k-skeleton* of Δ.

Maximal faces of a simplicial complex are called *facets*. If every facet of Δ has the same dimension, we say Δ is *pure*. If the facets of Δ are $F_1, \ldots, F_s$, we write $\Delta = \langle F_1, \ldots, F_s\rangle$. When Δ is a d-dimensional pure complex, we call the $(d-1)$–faces the *ridges* of Δ. Lastly, given a subset $A \subset V$ of the vertices of Δ, we write $\Delta_A = \{\sigma\colon \sigma \in \Delta \quad \text{and} \quad \sigma \subset A\}$ for the *induced subcomplex* on vertex set A.

For a d-dimensional simplicial complex Δ, we denote by C_i the free abelian group generated by the i-faces of Δ. For $i \geq 0$, the *i-th boundary map* of Δ is the group homomorphism $\partial_i : C_i \to C_{i-1}$ defined on an i-face $\sigma = \{j_1, \ldots, j_{i+1}\}$ with $j_1 < \cdots < j_{i+1}$ by

$$\partial_i(\sigma) = \sum_{k=1}^{i+1} (-1)^{k+1} (\sigma \setminus \{i_k\}).$$

An element $v \in C_i$ is called an i-chain of Δ. If $v \in \ker\partial_i$ we say v is a i-cycle, and finally if $v \in \operatorname{im}\partial_i$ we say v is an i-boundary.

Lemma 2.1.2. *Given a simplicial complex Δ, we have $\partial_i \circ \partial_{i+1} = 0$ for every i.*

Proof. See [80, Lemma 2.1]. □

A sequence of maps $f_1, f_2, \ldots, f_s$ such that $f_i \circ f_{i+1} = 0$ as in Lemma 2.1.2 is called a *chain complex*, and is often denoted by a series of arrows representing each map:

$$\cdots \longrightarrow C_i \longrightarrow C_{i-1} \longrightarrow C_{i-2} \longrightarrow \ldots.$$

A consequence of a sequence of maps $f_\bullet$ defining a chain complex is that $\operatorname{im} f_{i+1} \subset \ker f_i$. This containment allows us to define the *i-th homology groups* of the chain complex, which is the quotient $\frac{\ker f_i}{\operatorname{im} f_{i+1}}$. A chain complex such that every homology is zero is called *exact*.

The *i-th (reduced) homology group* of a simplicial complex Δ with boundary maps given by $\partial_\bullet$ is the quotient

$$\tilde{H}_i(\Delta; \mathbb{Z}) := \frac{\ker\partial_i}{\operatorname{im}\partial_{i+1}}.$$

Instead of defining the C_i to be abelian groups, we could have also defined them to be vector spaces over some underlying field $\mathbb{K}$. In this case, we write

$$\tilde{H}_i(\Delta; \mathbb{K}) := \frac{\ker \partial_i}{\operatorname{im} \partial_{i+1}}$$

for the *i-th homology group of Δ with coefficients over* $\mathbb{K}$. Algebraically, taking coefficients over some field $\mathbb{K}$ is equivalent to taking a tensor product of the C_i with $\mathbb{K}$. Note that homology groups may depend on the characteristic of the underlying field. When $\tilde{H}_i(\Delta; \mathbb{Z})$ has a subgroup with p-torsion (that is, $pa = 0$ for some nonzero $a \in \tilde{H}_i(\Delta; \mathbb{Z})$), we say Δ has p-torsion.

Example 2.1.3 (*Moore spaces and torsion*)**.** Given an abelian group G and an integer i, a simplicial complex Δ is called a *Moore space of G* if $\tilde{H}_i(\Delta; \mathbb{Z}) \cong G$ and $\tilde{H}_j(\Delta; \mathbb{Z}) = 0$ for $j \neq i$ (see e.g., [80, Chapter 2] for more information on Moore spaces). Figure 2.1 shows an example of a Moore space Δ for $\mathbb{Z}_3$ and $i = 1$. Its unique nontrivial homology group for $i > 0$ is $\tilde{H}_1(\Delta; \mathbb{Z}) \cong \mathbb{Z}_3$. Note that when we take coefficients over a field $\mathbb{K}$ there are two cases:

$$\tilde{H}_1(\Delta; \mathbb{K}) = \begin{cases} \mathbb{K} & \text{if} \quad \mathbb{K} \text{ is a field of characteristic } 3 \\ 0 & \text{otherwise.} \end{cases}$$

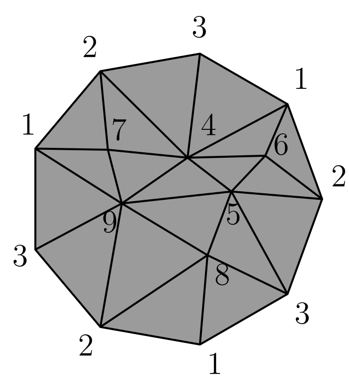


Figure 2.1: A Moore space for $\mathbb{Z}_3$, where shaded triangles correspond to facets of the simplicial complex.

Remark 2.1.4 (Void vs irrelevant complexes). There are two different notions of what an "empty" complex should be. On one hand, the *void complex* is defined by the empty collection $\{\}$ of subsets of a vertex set V. On the other hand, the *irrelevant complex* is given by the collection $\{\{\}\}$ that only contains the set given by the empty set. Although it may not seem like a big difference, following our definitions the void complex has dimension $-\infty$, while the irrelevant complex is -1-dimensional. Moreover, if we denote the void complex by Δ and the irrelevant complex by Γ, then we have $\tilde{H}_i(\Delta;\mathbb{Z}) = 0$ for every i (since every C_i is 0) and $\tilde{H}_{-1}(\Gamma;\mathbb{Z}) = \mathbb{Z}$ and $\tilde{H}_i(\Gamma;\mathbb{Z}) = 0$ for $i \neq -1$.

A geometric simplex K of dimension d is the convex hull of $d+1$ affinely independent points $v_1,\ldots,v_{d+1}$ in $\mathbb{R}^D$, in other words, the set $\{t_1v_1 + \cdots + t_{d+1}v_{d+1} \colon \sum t_i = 1 \quad \text{and} \quad t_i \geq 0\}$. The elements $v_1,\ldots,v_{d+1}$ are called the vertices of K. A subsimplex L of a (geometric) simplex K is called a *face* of K. A *geometric simplicial complex* Δ is a nonempty family of simplices such that:

1. Each face of any simplex Δ is also a simplex of Δ
2. The intersection $K_1 \cap K_2$ of any two simplices $K_1, K_2 \in \Delta$ is a face of both K_1 and K_2.

To every geometric simplicial complex Γ there is an abstract simplicial complex Δ, where if K is a simplex of Γ with vertices $v_1,\ldots,v_{d+1}$, then the set $\{v_1,\ldots,v_{d+1}\}$ is a face of Δ. In this case, the geometric complex Γ is called a *geometric realization* of Δ. A geometric realization of Δ will be denoted by $|\Delta|$.

It turns out that both notions we introduced so far of simplicial complexes are actually equivalent. We refer the reader to [133, Chapter 1] for more details on this equivalence. Here, we just state the following result:

Theorem 2.1.5 (**Geometric realization theorem).** *Let $d \geq 0$. Every finite d-dimensional simplicial complex Δ has a geometric realization in $\mathbb{R}^{2d+1}$.*

Proof. See [133, Theorem 1.6.1]. □

We are now ready to define the notion of a triangulation of a topological space. This concept will appear several times throughout this text.

Definition 2.1.6 (**Triangulations).** A simplicial complex Δ is a *triangulation* of a topological space X if $|\Delta|$ is homeomorphic to X.

If $X = S^d$ (a d-dimensional sphere) and Δ is a triangulation of X, we say Δ is a *simplicial sphere*. If X is a d-dimensional ball and Δ is a triangulation of X, we say Δ is a *simplicial ball*.

It is a standard fact that if Δ is a d-dimensional simplicial sphere, the homology groups of Δ are given by:

$$\tilde{H}_i(\Delta;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if} \quad i = d \\ 0 & \text{otherwise.} \end{cases}$$

For a simplicial complex Δ and a face σ of Δ, we define the *link* and *star* of σ in Δ as follows:

$$\mathrm{link}_\Delta(\sigma) := \{\tau \colon \tau \cup \sigma \in \Delta \quad \text{and} \quad \tau \cap \sigma = \emptyset\} \quad \text{and} \quad \mathrm{star}_\Delta(\sigma) := \{\tau \colon \tau \cup \sigma \in \Delta\}.$$

Moreover, given a vertex v of Δ, the *deletion* of v in Δ is the complex $\mathrm{del}_\Delta(v) := \{\sigma \in \Delta \colon v \notin \sigma\}$.

The link and star of a face of Δ should be seen as subcomplexes of Δ containing the local information of Δ.

An important tool in the study of homology of simplicial complexes is the *Mayer-Vietoris long exact sequence*, which relates the homology groups of a space $X = A \cup B$, with the homology groups of the intersection $A \cap B$ and the homology groups of A and B separately. Below we state the version of this long exact sequence that we will need.

Theorem 2.1.7 (**Mayer-Vietoris long exact sequence for** link **and** del)**.** *Let Δ be a simplicial complex and a vertex v of Δ. Then we have the following long exact sequence:*

$$\cdots \to \tilde{H}_i(\mathrm{link}_\Delta(v);\mathbb{K}) \to \tilde{H}_i(\mathrm{del}_\Delta(v);\mathbb{K}) \to \tilde{H}_i(\Delta;\mathbb{K}) \to \tilde{H}_{i-1}(\mathrm{link}_\Delta(v);\mathbb{K}) \to \cdots$$

Proof. The long exact sequence above follows from the standard Mayer-Vietoris sequence (see [80, Chapter 2]), by taking $\Delta = A \cup B$ where $A = \mathrm{del}_\Delta(v)$, $B = \mathrm{star}_\Delta(v)$ and noticing that $\mathrm{link}_\Delta(v) = \mathrm{del}_\Delta(v) \cap \mathrm{star}_\Delta(v)$, and $\tilde{H}_i(\mathrm{star}_\Delta(v);\mathbb{K}) = 0$ for every i. $\square$

The last topological definition that we will need is the notion of homotopy equivalence. Two continuous maps $f, g : X \to Y$ between topological spaces are said to be *homotopic* if there exists a *homotopy* between them, that is, a map

$$H : X \times [0,1] \to Y \quad \text{where} \quad H(x,0) = f(x) \quad \text{and} \quad H(x,1) = g(x).$$

A continuous map $f : X \to Y$ is called a *homotopy equivalence* if there exists a map $g : Y \to X$ such that $f \circ g$ is homotopic to the identity in Y, and $g \circ f$ is homotopic to the identity in X. In this case we say X and Y are homotopy equivalent, and we write $X \cong Y$. If X is homotopy equivalent to a point, X is said to be *contractible*.

Remark 2.1.8 (Homotopy groups and general facts). Just as we defined homology groups of a space X, we could have defined the *i-th homotopy groups* $\pi_i(X; x_0)$ (for $i > 0$ and some $x_0 \in X$) of a space X. The elements of $\pi_i(X, x_0)$ are homotopy classes of maps $f : S^i \to X$ such that $f(s_0) = x_0$ for some fixed $s_0 \in S^i$, and the operation is defined by "concatenation" of such maps (see [80, Chapter 1 and Chapter 4] for more details). A path-connected space X such that $\pi_1(X; x_0) = 0$ for some (equivalently, any) x_0 is called *simply connected*.

Throughout the text, we will not need results about homotopy groups. In order to understand where some definitions (and results) are coming from, however, it is very useful to know the following facts:

1. (*Hurewicz' theorem* [80, Theorem 2A.1]) If X is path-connected, the abelian group $\tilde{H}_1(X; \mathbb{Z})$ is isomorphic to the abelianization of the *fundamental group* $\pi_1(X; x_0)$ of X (for any x_0).

   In particular, since there are nonabelian groups such that their abelianization is trivial, it is possible for a space X to have $\tilde{H}_1(X; \mathbb{Z}) = 0$ and $\pi_1(X; x_0) \neq 0$. The most famous (simplicial) example is a triangulation of the *Poincaré sphere*, which has fundamental group isomorphic to the alternating group on 5 elements A_5, and hence by Hurewicz' theorem, it has trivial first homology group.

2. (*A criterion for homotopy equivalence to a sphere* [14, Fact 9.18]) If Δ is a simply connected simplicial complex such that $\tilde{H}_i(\Delta; \mathbb{Z}) = 0$ for every $i \neq k$ and $\tilde{H}_k(\Delta; \mathbb{Z}) = \mathbb{Z}$, then Δ is homotopy equivalent to S^k.

3. (*Homology vs homotopy* [80])[1] Let X and Y be two spaces.

   (a) X and Y can have the same homology groups, but different homotopy groups

   (b) X and Y can have the same homotopy groups, but different homology groups

Remark 2.1.9 (Useful sources of examples). Finding examples of complexes in certain classes introduced in this section can be a very hard task. For example, how does one find a triangulation of a manifold that isn't a sphere, or of a homology sphere that isn't a simplicial sphere? Below we list resources that are extremely useful for finding interesting examples:

1. Hachimori's library of simplicial complexes [77]

2. The manifold page by Frank Lutz [126]

3. The SimplicialComplexes package in Macaulay2 [179, 180]

4. The built in examples in SageMath [195]

2.2 Special classes of complexes

We now introduce the classes of complexes and special properties of simplicial complexes that we will use throughout the text. These properties will often be very useful for topological, combinatorial or algebraic reasons. We note that several of the concepts introduced in this section have analogues in algebra and graph theory. Such equivalences will be made clear in future sections.

A d-dimensional pure simplicial complex $\Delta = \langle F_1, \dots, F_s \rangle$ is said to be *strongly connected* if for every pair G_1, G_2 of facets of Δ, there exists a sequence of facets $F_{i_1}, \dots, F_{i_j}$ such that $G_1 = F_1$, $G_2 = F_{i_j}$ and $|F_{i_k} \cap F_{i_{k+1}}| = d$ for every $k = 1, \dots, j-1$.

Definition 2.2.1 (Pseudomanifolds (with and without boundary) and orientability). A simplicial complex is a *pseudomanifold* if the following conditions are satisfied:

[1]We note that given a map $f : X \to Y$, it induces a map from homotopy groups of X to homotopy groups of Y. If the induced maps are all isomorphisms, f is called a *weak homotopy equivalence*. It is well-known that weak homotopy equivalences also induce isomorphisms for homology and cohomology groups (see [80, Proposition 4.21]). In particular, [80, Proposition 4.21] is the result that explains the common intuition that homotopy is a finer invariant than homology.

The two items should be viewed as the consequence of two spaces having isomorphic homotopy (resp. homology) groups but the isomorphisms do not arise from a weak homotopy equivalence.

1. Δ is pure,

2. Δ is strongly connected,

3. every ridge τ of Δ is contained in at most two facets.

A ridge of a pseudomanifold Δ is called a *boundary ridge* if τ is only contained in a single facet. Let $\tau_1, \ldots, \tau_r$ be the boundary ridges of Δ. The *boundary* of Δ is the simplicial complex $\partial\Delta = \langle \tau_1, \ldots, \tau_r \rangle$. If $\partial\Delta = \emptyset$ we say Δ is a pseudomanifold without boundary.

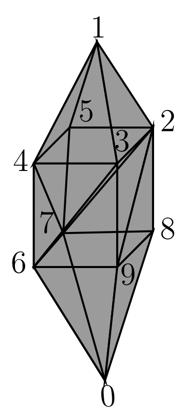
A pseudomanifold Δ without boundary is said to be *orientable over* $\mathbb{K}$ if $\tilde{H}_d(\Delta; \mathbb{K}) = \mathbb{K}$. In this case, a generator ε of $\tilde{H}_d(\Delta; \mathbb{K})$ is called an *orientation* of Δ.[2]

The conditions that a pseudomanifold Δ is pure, and that every ridge of Δ is contained in at most 2 facets of Δ should be compared to the condition that a manifold is locally homeomorphic to an euclidean space of the appropriate dimension. More specifically, if Δ triangulates a manifold, these two conditions must also be satisfied. The class of pseudomanifolds is a lot larger than the class of triangulations of manifolds.

Example 2.2.2 (The pinched torus: a "bad" orientable pseudomanifold). Let

$$\Lambda = \langle 123, 134, 145, 125, 239, 289, 257, 278, 457, 674, 346, 369, 781, 671, 691, 891 \rangle$$

be the simplicial complex obtained by identifying vertices 0 and 1 in the simplicial complex Γ below. The homology groups of Λ are: $\tilde{H}_0(\Lambda; \mathbb{Z}) = 0$, $\tilde{H}_1(\Lambda; \mathbb{Z}) = \mathbb{Z}$ and $\tilde{H}_2(\Lambda; \mathbb{Z}) = \mathbb{Z}$



[2] It is also possible to define orientability for pseudomanifolds with boundary, by asking that the top last relative homology of Δ and $\partial\Delta$ is $\mathbb{Z}$. Moreover, in the definition of orientable pseudomanifolds without boundary, it is enough to say $\tilde{H}_d(\Delta; \mathbb{K}) \neq 0$.

The complex Λ is a triangulation of the *pinched torus*, it is an example of a pseudomanifold that is not a triangulation of a manifold. The complex Γ above is an example of a simplicial sphere.

The next class of complexes that we will define is the class of *Cohen-Macaulay* complexes. The notion of a Cohen-Macaulay complex was introduced by Stanley in [181], who used a famous criterion due to Reisner [171] which classified the Cohen-Macaulay property of certain rings, which we will introduce in later sections of this chapter. Originally, the notion of Cohen-Macaulayness was a purely algebraic concept, but since Stanley introduced it to combinatorics, it now plays a major role in questions about combinatorics and topology[3].

Definition 2.2.3 (Cohen-Macaulay complexes, [171]**).** A d-dimensional complex Δ is *Cohen-Macaulay* (CM for short) over a field $\mathbb{K}$ if $\tilde{H}_i(\Delta;\mathbb{K}) = 0$ for $i < d$ and $\tilde{H}_j(\mathrm{link}_\Delta(\sigma);\mathbb{K}) = 0$ for every $j < \dim \mathrm{link}_\Delta(\sigma)$.

We are now ready to define the main class of simplicial complexes that we will use throughout this text.

Definition 2.2.4 (Homology spheres (or Gorenstein* complexes)). A d-dimensional simplicial complex Δ is called a *homology sphere* or equivalently a *Gorenstein** complex over a field $\mathbb{K}$ if

$$\tilde{H}_i(\mathrm{link}_\Delta(\sigma);\mathbb{K}) = \begin{cases} \mathbb{K} & \text{if} \quad i = d - |\sigma| - 1 \\ 0 & \text{otherwise.} \end{cases}$$

If Δ has a face τ such that $\sigma \cup \tau \in \Delta$ for every $\sigma \in \Delta$ (in other words, Δ is a cone with apex τ), and $\mathrm{link}_\Delta(\sigma)$ is a Gorenstein* complex, then we say Δ is a Gorenstein complex.

Next, we introduce the class of *shellable* complexes. Shellability was introduced by Bruggesser and Mani in 1970 [25] after being assumed in earlier proofs of the Euler-Poincaré formula for convex polytopes for example by Schläfi in 1852 [174]. We refer the reader to [208] for more historical details.

[3] In [15] Björner writes: “Much of the appeal of the concept of CM-ness stems from its ’interdisciplinary’ character, building on and having bearing on several mathematical areas in addition to combinatorics, notably several parts of algebra, geometry and topology. However, what bestows lasting importance on the concept is its remarkable record of being a key ingredient both for ’abstract’ theoretical understanding and for ’concrete’ problem solving in several of these diverse areas”

Definition 2.2.5 (Shellable complexes). A pure d-dimensional simplicial complex Δ is *shellable* if there exists an ordering $F_1, \ldots, F_s$ of the facets of Δ such that for every $k = 2, 3, \ldots, s$ the simplicial complex

$$\langle F_1, \ldots, F_{k-1} \rangle \cap \langle F_k \rangle$$

is pure of dimension $d - 1$.

The last class of complexes we introduce in this section is the class of *vertex decomposable* complexes, introduced by Provan and Billera in 1980 [166]. This property has by now been extended to a more algebraic/geometric context by Knutson, Miller and Yong [112] with the notion of geometric vertex decomposable ideals. Throughout this text we will not really need the notion of vertex decomposability, but we mention it here due to its similarity with the class of spherical complexes that will be introduced in later chapters.

Definition 2.2.6 (Vertex decomposable complexes). A pure simplicial complex is *vertex decomposable* if either

1. $\Delta = \{\{\}\}$, or Δ is a simplex, or
2. there exists a vertex v of Δ such that every facet of $\mathrm{del}_\Delta(v)$ is a facet of Δ, and $\mathrm{link}_\Delta(v)$ and $\mathrm{del}_\Delta(v)$ are vertex decomposable.

Below we include some of the known relationships between the classes of complexes introduced in this section. All implications are strict.

1. Vertex decomposable $\implies$ Shellable $\implies$ Cohen-Macaulay
2. Simplicial sphere $\implies$ Homology sphere $\implies$ Cohen-Macaulay
3. Shellable and pseudomanifold $\implies$ Simplicial sphere or ball

Remark 2.2.7 (The many different notions of a "sphere"). One of the implications mentioned above says that not every homology sphere is a simplicial sphere, which was also mentioned in Remark 2.1.8. There is another important property of spheres that we have not introduced: *polytopality*.

A *polytope* is the convex hull of a finite set of points in euclidean space. We call the points the *vertices* of the polytope. A subset σ of a polytope $P \subset \mathbb{R}^n$ is called a *face* of P if there exists a vector $\mathbf{w} \in \mathbb{R}^n$ such that the points of P that maximize the dot product with

$\mathbf{w}$ are exactly the points in σ. If every proper face of P (i.e. the face is not P itself) is a geometric simplex, we say P is a *simplicial polytope*, and the boundary of P is a simplicial complex called the *boundary complex* of P. A simplicial sphere Δ is said to be *polytopal* if Δ can be realized as the boundary complex of a simplicial polytope.

At first, it might seem like every simplicial sphere is polytopal. As it turns out, not only is this intuition wrong, it is in fact *very wrong*[4]. In 2005, Björner, Paffenholz, Sjöstrand and Ziegler [16, Section 6.1] show that there are at least

$$\frac{2^{2^n/\sqrt{n}}}{(\frac{2n}{e})^{2n}}$$

simplicial spheres. On the other hand, Alon showed in 1986 [6, Theorem 5.1] (see also [70]) that there are at most

$$2^{8n^3+O(n^2)}$$

simplicial polytopes on $2n$ vertices. And so *most spheres are nonpolytopal*.

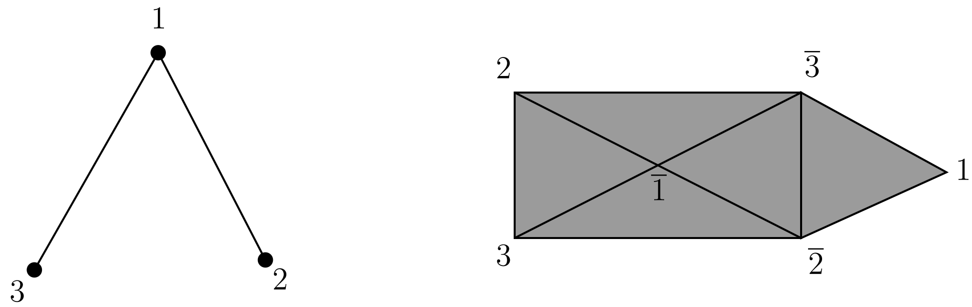
As was mentioned before, shellable complexes were originally introduced in the context of simplicial polytopes (or equivalently, polytopal spheres). Not every simplicial sphere is shellable. To the best of the author's knowledge, given d and n, the behaviour of the ratio between the number of combinatorially distinct d-spheres on n vertices and combinatorially distinct shellable spheres is unknown as n increases (for more information on the topic, see for example [204]).

We note that the observation by Björner et. al. was already known by Kalai in his paper "Many triangulated spheres" from 1988 [104]. The advantage of the approach from [16] to our goals, is that the spheres they use to get their bounds are useful to us.

We now introduce the final prerequisites to state the construction mentioned in Remark 2.2.7, which is (in this setting) originally due to unpublished work of Bier in 1992, but as we will see in Section 2.3 and Example 2.5.7 it has appeared before in graph theory and commutative algebra.

---

[4]The question of whether a given homology sphere is polytopal is called the *Simplicial Steinitz problem* and has a fascinating and rich history. This is another area of geometric combinatorics where commutative algebra ends up playing a key role in many of the major related results.



Δ The Bier ball of Δ

Figure 2.2: The complex $\Delta = \langle \{1, 2\}, \{1, 3\} \rangle$ and its Bier ball. The Bier sphere of Δ is then easily seen to be the pentagon given by $\partial B(\Delta) = \langle \{1, \bar{2}\}, \{3, \bar{2}\}, \{2, 3\}, \{2, \bar{3}\}, \{1, \bar{3}\} \rangle$

Definition 2.2.8 (Bier balls and spheres[5]). Let Δ be a simplicial complex on vertex set $V = \{v_1, \ldots, v_n\}$. Then simplicial complex $B(\Delta)$ on vertex set $\{v_1, \ldots, v_n, \bar{v}_1, \ldots, \bar{v}_n\}$ and facets:

$$\{v \colon v \in F\} \cup \{\bar{v} \colon v \notin F\} \quad \text{for every} \quad \text{face } F \text{ of } \Delta$$

is called the *Bier ball* of Δ if $\partial B(\Delta) \neq \emptyset$ and the *Bier sphere* of Δ if $\partial B(\Delta) = \emptyset$. If $\partial B(\Delta) \neq \emptyset$ we say the complex $\partial B(\Delta)$ is the Bier sphere of Δ.

The following remarks should clarify the different cases of Bier balls and spheres.

1. Every Bier ball is a simplicial ball, and every Bier sphere is a simplicial sphere.

2. When $B(\Delta)$ is already a sphere, the complex Δ does not have a Bier ball. Such complexes are easily classified: they are exactly the complexes Δ such that the nonfaces of $B(\Delta)$ are disjoint sets of size 2. We will give a simple algebraic proof that these complexes are not balls in the end of Section 2.5.

2.3 Graph theory and independence complexes

Recall from Section 2.1 that a graph $G = (V, E)$ is a tuple of *vertices* and *edges*. A subset A of vertices of a graph G is said to be *independent* if the induced subgraph G_A consists of

[5] The original definition of Bier spheres due to Bier used the notion of Alexander duality and deleted join. The Alexander dual of Δ is the simplicial complex $\Delta^\vee$ where the faces are the complements of nonfaces of Δ.

isolated vertices, in other words, there is no edge between vertices of A. In 1983 Gutman and Harary [76] introduced the *independence polynomial* of a graph G, defined as

$$I(x,G) = i_0 + i_1x + \cdots + i_sx^s$$

where i_k denotes the number of independent sets of size k of G (in particular $i_0 = 1$). The sequence $i_0, \ldots, i_s$ is called the *independent set sequence* of G. The original motivation for studying the independence polynomial of a graph comes from the fact that the independence polynomial generalizes the *matching polynomial* of G, where instead of counting the number of independent sets of vertices of a graph G, the matching polynomial of G counts the number of independent edges (i.e. edges that do not share a vertex). The matching polynomial of G is (by definition) the independence polynomial of a graph often called the *line graph* of G, and so the independence polynomial is a natural generalization of the matching polynomial.

A sequence of numbers $a_0, \ldots, a_s$ is said to be *unimodal* if there exists k such that

$$a_0 \leq \cdots \leq a_k \geq \cdots \geq a_s.$$

In 1987, Alavi, Malde, Schwenk and Erdős [5] considered a question asked by Wilf [6] of whether the independent set sequence of every graph G is unimodal. It should be noted that at the time, this question would have been a perfectly reasonable question, considering it was shown in [175] [7] by Schwenk that the coefficients of the matching polynomial of any graph G formed a unimodal sequence [8]. We will come back to the results in [5] in later chapters.

Given an independent set A of a graph G, every subset of A is also an independent set of G. In particular, the *independence complex* $\mathrm{Ind}(G)$ of a graph G is the simplicial complex on the same vertex set as G, and $\tau \in \mathrm{Ind}(G)$ if and only if τ is an independent set

[6]The original mention that this question was asked by H. Wilf is in [5], where the authors write: "He seemed to be somewhat sceptical of this conjectured unimodality. We shall not only show that the unimodal conjecture is false, but that, unlike the edge sequence, the numbers in the vertex independence sequence are totally unconstrained". We could not find any publication by H. Wilf where this question is formally asked.

[7]In the same paper, Schwenk suggests his approach could be applied to prove a conjecture of Read [168] from 1968, that says the coefficients of the chromatic polynomial of a graph form a unimodal sequence. Read's conjecture was only proven in 2012 by June Huh [97] using deep algebraic geometry techniques.

[8]See also [82], where Heilmann and Lieb show (independently) the stronger result that the matching polynomial of a graph is real rooted.

of G. A simplicial complex Δ is called a *flag complex* if there exists a graph G such that $\Delta = \mathrm{Ind}(G)$.

The notion of independence complex may be easily generalized to arbitray simplicial complexes: the *independence complex* $\mathrm{Ind}(\Delta)$ of a simplicial complex Δ is the simplicial complex on the same vertex set as Δ, and σ is a face of $\mathrm{Ind}(\Delta)$ if and only if σ does not contain any facet of Δ. This perspective of considering independent sets in a simplicial complex turns out to be extremely useful.

At this point, we finally introduce the deceptively simple object that is the motivation for most (if not all) results in this thesis.

Definition 2.3.1 (f-vectors of simplicial complexes). Given a d-dimensional simplicial complex Δ, the *f-vector* of Δ, denoted by $f(\Delta)$ is the vector $f(\Delta) = (f_{-1}, \ldots, f_d)$, where f_i is the number of i-dimensional faces of Δ. Similarly, the f-polynomial of Δ is the generating function $f_\Delta(x)$ of $f(\Delta)$. If it is clear from context, we ommit the Δ from the notation of the f-polynomial. Note that if $\Delta = \mathrm{Ind}(G)$ for some graph G, then $f_{k-1} = i_k$, where i_k is the number of independent sets of size k of G.

We now introduce the graph theoretic versions of many of the properties introduced in Sections 2.1 and 2.2

Definition 2.3.2 (Well-covered, very well-covered and 1-well-covered graphs). We say a graph G is:

1. *well-covered* if every maximal independent set of G has the same size;

2. *very well-covered* if G has $2n$ vertices, is well-covered and every maximal independent set of G has size n;

3. *1-well-covered* if G has at least two vertices and the graph $\mathrm{del}_G(v)$ is well-covered for every vertex v of G. If moreover G does not have isolated vertices, then G is called a *W_2 graph*.

The notion of well-covered graphs was introduced by Plummel in a survey paper from 1970 [163], where the author defines this property but only mentions that no work has been done in this class of graphs. In the same paper, Plummel mentions that finding well-covered graphs is an interesting problem. In the language of simplicial complexes, saying a graph G is well-covered is equivalent to saying $\mathrm{Ind}(G)$ is a pure simplicial complex.

The class of very well-covered graphs was introduced by Favaron in 1982 [58] when the author generalized an earlier result of Ravindra [167] which classified bipartite well-covered graphs. In [58] Favaron gives a structural characterization of very well-covered graphs that nowadays is often used in commutative algebra. A graph being very well-covered is equivalent to $\mathrm{Ind}(G)$ having $2n$ vertices, $\dim \mathrm{Ind}(G) = n-1$ and $\mathrm{Ind}(G)$ being pure.

Lastly, the class of 1-well-covered graphs was introduced by Staples in 1979 [189] [9] where she generalized natural properties that well-covered graphs satisfy, in order to go from what she calls W_1 graphs (well-covered) to W_n graphs. Here we only care about the class of W_2 graphs that she introduced. This class of graphs turns out to be extremely related to the notion of generalized homology spheres introduced in Section 2.2, as we will see in later sections.

Around the 1980s, the problem of finding special classes of well-covered graphs attracted a lot of interest. Since Plummel's paper in 1970 it was already known that a graph G being well-covered posed strong structural restrictions on G. Before stating the next result, we first need to introduce one last class of graphs that has appeared under several different names in the literature.

Definition 2.3.3 (Whiskered graphs, leafy extensions and corona products). Let G be a graph on vertex set $V = \{v_1, \ldots, v_n\}$ and edge set E. The *whiskering* (also known as *leafy extension* [23], or corona of G with a vertex [63]) $w(G)$ of G is the graph with vertex set $\{v_1, \ldots, v_n, \bar{v}_1, \ldots, \bar{v}_n\}$ and edge set

$$E \cup \{v_i\bar{v}_i \colon 1 \leq i \leq n\}.$$

Note that we can also define the whiskering of a simplicial complex by adding the same edges as the new facets of the whiskered complex.

Whiskered graphs play a key role in the structural restrictions imposed by a graph being well-covered. More concretely, in 1993 Finbow, Hartnell and Nowakowski showed the

[9] The name 1-well-covered graphs is not used in [189], but the diference between 1-well-covered graphs and the class of W_2 graphs is not relevant in most cases. The first time where the author could find an explicit mention of the term 1-well-covered graphs is in the work of Pinter [162] in 1992.

following result which generalized an earlier result of Finbow and Hartnell [60]:

Theorem 2.3.4 ([59, Corollary 5]). *Let G be a well-covered graph such that the smallest induced cycle of G has length $n \geq 6$. Then if G is not the complete graph K_1 or the cycle graph C_7, G must be a whiskered graph.*

At this point, the reader should compare Definitions 2.2.8 and 2.3.3. The similarity in notation for the sets of vertices of $B(\Delta)$ and $w(G)$ is not a coincidence. It was first noticed by Murai in [152] that

$$\mathrm{Ind}(w(\Delta)) = B(\Delta).$$

In other words, whiskering a complex corresponds to the same transformation as the construction introduced by Bier that outputs a simplicial ball (or sphere).

2.4 Hilbert series

We now begin to introduce the algebraic background required for the rest of this thesis. From now on, $\mathbb{K}$ will always denote a field.

A commutative ring R is said to be $\mathbb{N}^s$-*graded* if there exists a ring (usually a field) R_0 and R_0-modules $R_{\mathbf{a}}$ for $\mathbf{a} \in \mathbb{N}^s$ such that

$$R = \bigoplus_{\mathbf{a} \in \mathbb{N}^s} R_{\mathbf{a}}$$

and $R_{\mathbf{a}} R_{\mathbf{b}} \subset R_{\mathbf{a}+\mathbf{b}}$. In most cases we will consider, R will be a polynomial ring, $s = 1$, $R_0 = \mathbb{K}$ and the grading is given by the usual degree of a polynomial. When the degree of every variable in a polynomial ring R is 1, we say R is *standard graded*.

Similarly, when R is $\mathbb{N}^s$-graded, an R-module M is said to be $\mathbb{N}^s$-graded if there exists R_0-modules $M_{\mathbf{a}}$ for $\mathbf{a} \in \mathbb{N}^{\mathbf{s}}$ such that

$$M = \bigoplus_{\mathbf{a} \in \mathbb{N}^s} M_{\mathbf{a}}$$

and $R_{\mathbf{a}} M_{\mathbf{b}} \subset M_{\mathbf{a}+\mathbf{b}}$. When N is a graded submodule of M, the quotient M/N is also graded. Graded ideals are also called homogeneous ideals.

Let $R = \mathbb{K}[x_1, \ldots, x_n]$. The *Hilbert function* of a graded finitely generated R-module M is the numerical function $\mathrm{HF}_M : \mathbb{N} \to \mathbb{N}$ such that $\mathrm{HF}_M(n) = \dim M_n$, that is, the

Hilbert function of M keeps track of the dimension of each graded piece of M as a $\mathbb{K}$-vector space. The *Hilbert series* of M is the series

$$\mathrm{HS}_M(t) = \sum_{i=0}^{\infty} \mathrm{HF}_M(i)t^i.$$

In our setting, where $R = \mathbb{K}[x_1, \dots, x_n]$ with the standard grading, the Hilbert Series of any ideal I is a rational function of the form[10]

$$\mathrm{HS}_{R/I}(t) = \frac{h(t)}{(1-t)^d}. \tag{2.1}$$

The coefficients of the polynomial $h(t)$ form a vector called the *h-vector* of R/I. We now briefly introduce the necessary concepts that give meaning to the integer d in the formula above.

The *(Krull) dimension* of a ring R, denoted by $\dim R$, is the supremum of lengths r over all possible decreasing chains of prime ideals

$$\mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_r.$$

For an ideal I in a polynomial ring R over a field, the *height* of I is $n - \dim R/I$, where n is the number of variables of R. The integer d in (2.1) is the dimension of R/I. For a finitely generated R-module M, we set $\dim M = \dim R/\operatorname{Ann}(M)$, where $\operatorname{Ann}(M) = \{r \in R \colon rm = 0 \quad \text{for all} \quad m \in M\}$ is the ideal of elements in R that annihilate all of M.

One of the first natural questions that one can ask about Hilbert series is: *what sequences arise as the Hilbert series of R/I?*

In order to state the answer, we first need to introduce some definitions.

Definition 2.4.1. Let m and d be positive integers. The *d-binomial expansion* of m is the expression

$$m = \binom{a_d}{d} + \binom{a_{d-1}}{d-1} + \cdots + \binom{a_j}{j},$$

[10] In [127] Macaulay mentions that it was Ostrowski [157] who noticed it was possible to write the generating function of the Hilbert series in a ”neat” form.

where $a_d > a_{d-1} > \cdots > a_j > j \geq 1$. Moreover, let [11]

$$m^{(d)} = \binom{a_d + 1}{d+1} + \binom{a_{d-1} + 1}{d} + \cdots + \binom{a_j + 1}{j+1}.$$

In 1927 Macaulay [127] (see [27] for a modern treatment) Macaulay showed the following amazing result:

Theorem 2.4.2 ([127], [27, Theorem 4.2.10])**.** *A sequence* $(1, h_1, h_2, \dots)$ *is the sequence of values of the Hilbert function of some standard graded algebra* R/I*, where* $R = \mathbb{K}[x_1, \dots, x_n]$*,* $n = h_1$ *and* $\mathbb{K}$ *a field if and only if*

$$h_{i+1} \leq h_i^{(i)} \quad \textit{for all} \quad i \geq 0.$$

Sequences satisfying the conditions in Theorem 2.4.2 are called M*-sequences* or O*-sequences*. The study of O-sequences is extremely rich, but before stating some of the relevant results, we need to introduce special classes of algebras, which will be the topic of Section 2.5.

For now, we define the algebraic objects that allows us to study combinatorial an topological properties of simplicial complexes using commutative algebra.

Definition 2.4.3 (Stanley-Reisner ideals). Let Δ be a simplicial complex on vertex set V. The *Stanley-Reisner* ideal $I_\Delta \subset R = \mathbb{K}[x_v \colon v \in V]$ of Δ is the ideal of nonfaces:

$$I_\Delta = (x_\sigma \colon \sigma \notin \Delta) \quad \text{where} \quad x_\sigma = \prod_{v \in \sigma} x_v.$$

For a monomial $m = x_1^{a_1} \dots x_n^{a_n}$, the *support* of m is the set $\operatorname{supp}(m) = \{x_i \colon a_i > 0\}$. Note that if a monomial m is nonzero in R/I_Δ for some simplicial complex R, then the support of m must be a face of Δ. In particular, since monomials are linearly independent, one should expect a closed formula for the Hilbert series of R/I_Δ to involve the f-vector of Δ. In order to state the connection, we first define the second object that motivates most (if not all) the results in this thesis.

Definition 2.4.4 (h-vectors of simplicial complexes). Let Δ be a simplicial complex with f-vector $(1, f_0, \dots, f_d)$. The *h-vector* of Δ is the vector $h(\Delta) = (h_0, \dots, h_{d+1})$ defined by

[11] The integer $m^{(d)}$ is sometimes called a *pseudopower* of m

the following polynomial relations:

$$\sum_{i=0}^{d+1} f_{i-1}(x-1)^{d+1-i} = \sum_{i=0}^{d+1} h_i x^{d+1-i} \quad \text{or equivalently,} \quad h_i = \sum_{k=0}^{i} (-1)^{i-k} \binom{d+1-k}{i-k} f_{k-1}.$$

The h-*polynomial* $h_\Delta(t)$ of Δ is the generating function of $h(\Delta)$.

Before stating the connection between h-vectors and Hilbert series, we first give an example of a method of computing the h-vector often called *Stanley's trick*, and start describing the h-vector of a 2-dimensional simplicial sphere. The original classification of f-vectors (or equivalently, h-vectors) of 2-dimensional spheres was originally obtained by Steinitz [191] in 1906.

Example 2.4.5 (Some patterns: Stanley's trick and Euler's relation). In order to compute the h-vector of a simplicial complex Δ, we may do a procedure similar to filling Pascal's triangle. We begin by replacing the right hand side of the triangle with the f-vector of Δ (instead of all ones). We then fill in the rest of the entries by subtracting (instead of adding) two consecutive entries, taking the right-most term to be the positive term in this difference. After filling $d+3$ lines of this triangle, the entries of the last row are the entries of $h(\Delta)$.

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & v & & & \\
 & & 1 & & v-1 & & e & & \\
 & 1 & & v-2 & & e-v+1 & & f & \\
1 & & v-3 & & e-2v+3 & & f-e+v-1 & &
\end{array}$$

$$h(\Delta) = (1,\ v-3,\ e-2v+3,\ f-e+v-1)$$

Figure 2.3: Stanley's trick for a 2-dimensional complex with f-vector $(1, v, e, f)$.

Assume now that Δ is a 2-dimensional simplicial sphere[12]. The well-known Euler relation says $v - e + f = 2$, and so the h-vector of Δ is given by $h(\Delta) = (1, v-3, e-2v+3, 1)$. Note that once we apply Euler's relation, $h(\Delta)$ does not involve the number of triangles f of Δ. In particular, since the process of going from f- to h-vectors is invertible,

[12]A famous result of Steinitz says every 2-dimensional homology sphere is a polytopal simplicial sphere, see [208, Chapter 4 and pp. 142] for more details.

the h-vector allows us to give a (at this point, partial) characterization of the f-vectors of 2-dimensional spheres that only uses the number of vertices and edges. In Section 2.5 we will give the rest of the characterization.

We are now ready to state the connection between h-vectors and Hilbert series, first noticed by Stanley [181], that led to the first major applications of commutative algebra to the study of simplicial complexes.

Theorem 2.4.6. *Let Δ be a $(d-1)$–dimensional simplicial complex with Stanley-Reisner ideal $I_\Delta \subset R = \mathbb{K}[x_1, \ldots, x_n]$. Then*

$$\mathrm{HS}_{R/I_\Delta}(t) = \frac{h_\Delta(t)}{(1-t)^d}.$$

Proof. See [27, Chapter 5]. □

To finish this section, we state McMullen's g-conjecture as it will motivate the topics of the next section.

Definition 2.4.7 (**g-vectors of simplicial complexes** [13]**).** Let Δ be a $(d-1)$–dimensional simplicial complex with h-vector $(h_0, \ldots, h_d)$. The *g-vector* of Δ is the vector $g(\Delta) = (g_0, g_1, \ldots, g_{\lfloor d/2 \rfloor})$, where $g_0 = 1$ and $g_i = h_i - h_{i-1}$ for $i = 1, \ldots, \lfloor d/2 \rfloor$.

Conjecture 2.4.8 (McMullen's g-conjecture). *The vector $(h_0, \ldots, h_d)$ is the h-vector of a homology sphere if and only if*

1. $h_0 = 1$
2. $h_i = h_{d-i}$ *for* $0 \leq i \leq d$
3. *The vector $(g_0, \ldots, g_{\lfloor d/2 \rfloor})$, where $g_0 = 1$ and $g_i = h_i - h_{i-1}$ for $0 < i < \lfloor d/2 \rfloor$ satisfies*

$$g_{i+1} \leq g_i^{(i)} \quad \textit{for all} \quad i \geq 0,$$

or equivalently, the vector $(g_0, \ldots, g_{\lfloor d/2 \rfloor})$ is an O-sequence.

[13] In earlier papers, the letter g (with certain superscripts and subscripts) was used for entries of both the h-vector and the g-vector. The use of the letter h appears to have started after the connection with Hilbert series had been established. This would explain the reason we go from f to h and then back to g.

A naive strategy to prove the (now known in many cases, such as for simplicial spheres and polytopes [3,4,13,108,158,185]) g-conjecture, would be to find an algebraic construction that takes an algebra A with Hilbert series $\mathrm{HS}_A(t) = \frac{h(t)}{(1-t)^d}$, and outputs an algebra A' with Hilbert series $\mathrm{HS}_{A'}(t) = (1-t)\frac{h(t)}{(1-t)^d}$. If such a construction exists, at least for the face ring R/I_Δ of a homology $(d-1)$–sphere, and we apply it d times to R/I_Δ we would have a sequence of algebras $A_0 = R/I_\Delta, A_2, \ldots, A_d$ such that:

$$\mathrm{HS}_{A_i}(t) = (1-t)^i\,\mathrm{HS}_{R/I_\Delta} = (1-t)^i\frac{h_\Delta(t)}{(1-t)^d} \quad \text{for all} \quad 0 \leq i \leq d.$$

In particular, we would have $\mathrm{HS}_{A_d}(t) = h_\Delta(t)$. The construction must fail for A_d, since the polynomial $(1-t)h_\Delta(t)$ has at least one negative coefficient (the leading term). We may however ask that the construction works "when it makes sense", in other words, we want an algebra A_{d+1} such that $\mathrm{HF}_{A_{d+1}}(i) = g_i$, where $g_i = \max(h_i - h_{i-1}, 0)$ is the coefficient of degree i of the polynomial $(1-t)h_\Delta(t)$ after truncating it at the first negative coefficient.

Although at this point the suggested construction might seem ad hoc, it is actually extremely common and has been extensively studied in vastly different areas of mathematics for completely different purposes. As Hochster said in 1978 [90]: "life is really worth living" in rings where the first part of the construction works[14].

2.5 Translating the conditions from the g-conjecture to algebra

The goal of this section is to understand the construction mentioned at the end of Section 2.4 from a systematic perspective. This will lead us to the definition of Cohen-Macaulay and Gorenstein rings, as well as the notion of Lefschetz properties.

Remark 2.5.1. Standard graded vs Local rings Two of the main settings studied in commutative algebra are the classes of *local rings* (i.e. rings with a unique maximal ideal), and *standard graded rings*. While the exact statements of many of the results we will cite in this section are for local rings, it is known that general results about local rings should also hold (after translating the notions, for example, the unique maximal ideal of a local

[14]The original quote from [90, pp. 887] says: "Life is really worth living in a Noetherian ring R when all the local rings have the property that every s.o.p. is an R-sequence." We will see in the next section that this is essentially the condition we need to make sure the first part of the construction works.

ring becomes the unique maximal *homogeneous* ideal of the standard graded ring) for standard graded rings. For more details, we refer the reader to [27, Section 1.5], where the authors introduce **Local rings* as the analogue of local rings.

2.5.1 Cohen-Macaulay rings

Let R be a standard graded ring, M an R-module and $f \in R$ a homogeneous polynomial of degree a. Then we always have the exact sequence

$$M \xrightarrow{\times f} M \to M/fM \to 0.$$

Where the map $\times f$ shifts the degree by a, that is, it maps M_i to M_{i+a}. If the multiplication by f is injective, then we have the short exact sequence

$$0 \to M \xrightarrow{\times f} M \to M/fM \to 0,$$

and in particular, $\mathrm{HF}_{M/fM}(n) = \mathrm{HF}_M(n) - \mathrm{HF}_M(n-a)$ for every $n \geq 0$, which we can rewrite in a more concise form as:

$$\mathrm{HS}_{M/fM}(t) = (1 - t^a)\,\mathrm{HS}_M(t).$$

Note that the case $a = 1$ is exactly the first step in the first part of the construction at the end of Section 2.4.

Definition 2.5.2 (Regular sequences and depth). Let R be a ring and M an R-module. An element $f \in R$ is said to be M*-regular* if the multiplication map $\times f : M \to M$ is injective. A sequence of elements $\mathbf{f} = f_1, \dots, f_s$ is called an M*-regular sequence* if

1. The maps $\times f_i : M/(f_1, \dots, f_{i-1})M \to M/(f_1, \dots, f_{i-1})M$ are injective for all $i \leq s$
2. $M/\mathbf{f}M \neq 0$.

When $R = \mathbb{K}[x_1, \dots, x_n]$ is a polynomial ring over a field $\mathbb{K}$, and M a finitely generated R-module, the *depth* of M is

$$\operatorname{depth} M = \sup(s\colon \text{there exists a maximal } M\text{-regular sequence of } s \text{ elements contained in } \mathfrak{m})$$

where $\mathfrak{m} = (x_1, \ldots, x_n)$.

The following statement is a very useful property of the notion of depth.

Proposition 2.5.3. *Let $R = \mathbb{K}[x_1, \ldots, x_n]$, and M a finitely generated R-module, then*

$$\operatorname{depth} M \leq \dim M.$$

Proof. See [27, Proposition 1.2.12]. □

In the language of regular sequences and depth, the first part of the construction from Section 2.4 means we are looking for an ideal I such that $\dim R/I = \operatorname{depth} R/I$, which by Proposition 2.5.3 means R/I has maximal depth.

Definition 2.5.4 (Cohen-Macaulay rings). A ring R is called a *Cohen-Macaulay* (CM for short) ring if

$$\operatorname{depth} R = \dim R.$$

The name Cohen-Macaulay comes from a result due to Macaulay (for polynomial rings) from 1916 [128] and due to Cohen (for regular local rings) from 1946 [29, Theorem 21] called the *unmixed theorem*, which is an equivalent definition of Cohen-Macaulay rings that we will not need. Cohen-Macaulay rings are the most well-studied class of commutative rings, and this is because they appear frequently in different areas of mathematics, and end up being "the nice case" in very different areas of mathematics.

In algebraic geometry for example, if the localization of the coordinate ring of a variety V at a singular point is a Cohen-Macaulay ring, many useful theorems still hold, as if the point was not singular. This is because many results about the class of rings that correspond to smooth points, the class of regular rings (over perfect fields), also hold for Cohen-Macaulay rings. In combinatorics and topology, which is the relevant setting to us, the Stanley-Reisner ring of many interesting objects turn out to be Cohen-Macaulay.

The following statement relating the Cohen-Macaulay property of complexes (Definition 2.2.3) to the Cohen-Macaulay property of rings might seem tautological, but this is only because we introduced Cohen-Macaulay complexes before introducing Cohen-Macaulay rings. Historically, Cohen-Macaulay complexes were only considered after the following was shown by Reisner [171].

Theorem 2.5.5 (**Reisner's criterion,** [171]). *Let Δ be a simplicial complex and $I_\Delta \subset R = \mathbb{K}[x_1, \ldots, x_n]$ its Stanley-Reisner ideal. The following are equivalent:*

1. *R/I_Δ is a Cohen-Macaulay ring*

2. *Δ is Cohen-Macaulay over $\mathbb{K}$*

3. *$\tilde{H}_j(\mathrm{link}_\Delta(\sigma); \mathbb{K}) = 0$ for every $\sigma \in \Delta$ and every $j < \dim \mathrm{link}_\Delta(\sigma)$.*

The following statement is the main point in Stanley's proof of the Upper Bound Theorem for spheres.

Theorem 2.5.6 ([182, pp. 7]). *Let Δ be a triangulation of a d-dimensional manifold such that $\tilde{H}_i(\Delta; \mathbb{K}) = 0$ for all $i < d$. Then R/I_Δ is Cohen-Macaulay, where the underlying field of R is $\mathbb{K}$. In particular, the statement holds if Δ is a simplicial ball or sphere.*

Before moving on to the next class of rings, we first give an example which connects Definition 2.2.8 and Definition 2.3.3.

Example 2.5.7 (**Villarreal's interpretation of whiskering and Bier balls** [197]). Let $I(G) \subset \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_n]$[15] be the Stanley-Reisner ideal of the independence complex of a whiskered graph G, where the variable x_i corresponds to the vertex i, and the variable y_i corresponds to the whisker of i.

In 1990 Villarreal [197] noticed that the sequence of elements $x_1 - y_1, \ldots, x_n - y_n$ forms a regular sequence for $R/I(G)$, and in particular, since $\dim R/I(G) = n$, the ring $R/I(G)$ is Cohen-Macaulay. This result, which was 2 years earlier than the unpublished work of Bier, shows that the whiskering construction outputs Cohen-Macaulay complexes. Bier's result then showed the stronger statement that the whiskering construction outputs simplicial balls or spheres.

In a modern language, the main idea behind Villarreal's proof is that the ideal $I(G)$ is obtained by *polarizing* the ideal $J(G) = (x_1^2, \ldots, x_n^2) + (x_i x_j \colon ij \in E)$, where E is the edge set of G. More concretely, we have an isomorphism:

$$\frac{R}{I(G) + (x_1 - y_1, \ldots, x_n - y_n)} \cong \frac{\mathbb{K}[x_1, \ldots, x_n]}{J(G)},$$

[15]These ideals are called *edge ideals* and were introduced by Villarreal [197] in 1990 from a different perspective. We will introduce them properly in a later section.

and since the ring on the right hand side has dimension 0, it must also have depth 0 so both rings must be Cohen-Macaulay.

In 2005 Faridi [54] generalized Villarreal's approach by introducing the class of *grafted complexes*, which generalize the notion of whiskering by considering polarizations of monomial ideals I such that $\dim R/I = 0$. Faridi's main result is that the independence complexes of grafted complexes are Cohen-Macaulay.

In 2011 Murai [152] generalized Bier's result by independently studying the independence complexes of grafted complexes, and showing that they all correspond to simplicial balls or spheres, which he called *generalized Bier spheres*.

Even though all of the concepts above are about the same objects, none of the connections were known at the time the results were first obtained (starting already from the study of very well-covered graphs). In [35] in joint work with Susan Cooper, Sara Faridi, Lisa Nicklasson and Adam Van Tuyl, we make these connections concrete and prove new results regarding Cohen-Macaulay complexes arising from very well-covered graphs.

We note that we have not described all of the connections between the algebraic notion of polarization, the combinatorial notion of whiskering and Bier's topological construction. We refer the reader to [19, 35, 61] for more details.

2.5.2 Lefschetz properties

In the previous subsection we explored the algebraic notion of Cohen-Macaulayness, which can be described as the algebraic definition that guarantees the first part of the construction from the end of Section 2.4 works. We now focus on the algebraic notion that corresponds to the second part of the construction.

We begin by definining the sets of elements that allow us to do the first part of the construction in a Cohen-Macaulay ring. The main results of this thesis are about how to find special such sequences.

Definition 2.5.8 (Systems of parameters). Let R/I be a d-dimensional ring. A *homogeneous system of parameters* (s.o.p. for short) is a sequence $\theta_1, \ldots, \theta_d$ of elements in R such that $\dim \frac{R}{I+(\theta_1,\ldots,\theta_d)} = 0$. If the degree of θ_i is 1 for every i, we say further that the sequence $\theta_1, \ldots, \theta_d$ is a *linear* s.o.p. (l.s.o.p. for short).

When R/I is a Cohen-Macaulay ring, we have the following equivalence.

Proposition 2.5.9. *Let R/I be a d-dimensional Cohen-Macaulay ring. Then $\theta_1, \dots, \theta_d$ is a s.o.p. if and only if it is a regular sequence.*

Proof. See [27, Theorem 2.1.2]. $\square$

Once we have a 0-dimensional ring $A = R/I$ where $R = \mathbb{K}[x_1, \dots, x_n]$, it can be shown that A is in fact a finite dimensional vector space. Such rings are special examples of *artinian rings*. Since artinian rings are 0-dimensional, they do not admit any regular elements. This is easy to see, since A can be written as

$$A = A_0 \oplus \cdots \oplus A_d$$

for some d, given any element f we have $fg = 0$ for any $g \in A_d$ (so injectivity will never hold). The subspace $\operatorname{soc}(A) = \{g \in A \colon fg = 0 \quad \text{for every} \quad f \in \mathfrak{m}\}$, where $\mathfrak{m} = (x_1, \dots, x_n) \subset A$ is the maximal homogeneous ideal of A, is called the *socle* of A, and the minimum integer d such that $A_k = 0$ for $k > d$ is called the *socle degree* of A. As we have just mentioned, we always have $A_d \subseteq \operatorname{soc}(A)$, and when equality holds we say A is a *level algebra*.

Translating the second part of the construction from Section 2.4, we see that we want an element $\ell \in A_1$ such that the multiplication maps $\times\ell : A_i \to A_{i+1}$ are injective whenever they can be, and surjective otherwise. This is because in the exact sequence

$$A \xrightarrow{\times\ell} A \to A/\ell \to 0$$

if $\times\ell$ is injective in degree i, we have

$$0 \to A \xrightarrow{\times\ell} A \to A/\ell \to 0,$$

which implies $\mathrm{HF}_{A/\ell}(i) = \mathrm{HF}_A(i) - \mathrm{HF}_A(i-1)$, and if $\times\ell$ is surjective in degree i we have $(A/\ell)_i = 0$. Gathering these two cases, when such an element ℓ exists we have

$$\mathrm{HF}_{A/\ell}(i) = \begin{cases} \mathrm{HF}_A(i) - \mathrm{HF}_A(i-1) & \text{if} \quad \mathrm{HF}_A(i) \geq \mathrm{HF}_A(i-1) \text{ } (\times\ell \text{ is injective}) \\ 0 & \text{if} \quad \mathrm{HF}_A(i) \leq \mathrm{HF}_A(i-1) \text{ } (\times\ell \text{ is surjective}). \end{cases}$$

In summary, instead of asking for the element ℓ to be regular (which would never happen), we instead ask that *every multiplication map has full rank*, which is a very natural generalization of the notion of regular element.

Definition 2.5.10 (Lefschetz elements and Lefschetz properties). Let A be an artinian algebra of socle degree d over a field $\mathbb{K}$. A linear form ℓ is called a *weak Lefschetz element* if the multiplication maps $\times\ell : A_i \to A_{i+1}$ have full rank for every $0 \leq i \leq d-1$. A linear form ℓ is called a *strong Lefschetz element* if the multiplication maps $\times\ell^j : A_i \to A_{i+j}$ have full rank for every $i, j \geq 0$ and $i + j \leq d$.

If an algebra A has a weak (resp. strong) Lefschetz element we say the A satisfies (or has) the *weak Lefschetz property* (resp. *strong Lefschetz property*), which we abbreviate as WLP (resp. SLP).

The condition that defines a Lefschetz element is essentially a condition on the rank of a matrix, and hence it is reasonable to expect that if one Lefschetz element exists (over an infinite field), then infinitely many Lefschetz elements must exist. This idea is formalized in the following proposition.

Proposition 2.5.11. *Let A be an artinian algebra. The set of (weak or strong) Lefschetz elements of A forms an (possibly empty) open set in the Zariski topology.*

Proof. See [176, Exercise 4.4 and Remark 4.15]. □

The name "Lefschetz" comes from the well-known *hard Lefschetz theorem* from algebraic geometry (Theorem 1.1.4), which says if A is the cohomology ring of a smooth complex irreducible projective variety, then A has a strong Lefschetz element[16]. The study of Lefschetz properties in the sense of Definition 2.5.10 started when Stanley showed the following two results:

1. *The g-theorem holds for polytopal simplicial spheres* [185]. The proof uses a more general version of the hard Lefschetz theorem applied to a certain toric variety associated to the polytopal sphere. Even though these varieties are not smooth, the singularities are "mild" in the sense that a version of the hard Lefschetz theorem still holds.

[16] The theorem as stated says that A has a *strong Lefschetz element in the narrow sense*, and so A has the nSLP. These two notions are well-known to be equivalent in the setting of the theorem [78, pp. 104], so we do not make this distinction here.

2. *Monomial complete intersections satisfy the SLP* [186]. More concretely, Stanley showed that rings of the form $\frac{\mathbb{K}[x_1,\ldots,x_n]}{(x_1^{a_1},\ldots,x_n^{a_n})}$ satisfy the SLP by applying a version of the hard Lefschetz theorem to the product of projective spaces of the correct dimension.

One of the (combinatorial) motivations for Stanley's result on monomial complete intersections is the conclusions that follow in terms of inequalities the Hilbert series of a ring satisfying the SLP (or WLP) must satisfy. Here we state the most well-known consequence of the WLP: unimodality.

Proposition 2.5.12 (WLP and unimodality). *Let A be a standard graded artinian algebra satisfying the WLP and let $h_0, \ldots, h_d$ be the h-vector of A. Then there exists k such that*

$$h_0 \leq \cdots \leq h_k \geq \cdots \geq h_d.$$

Proof. Let ℓ be a weak Lefschetz element of A. Then we know every map $\times\ell : A_i \to A_{i+1}$ has full rank. Let k be the first index such that $\times\ell : A_k \to A_{k+1}$ is surjective. We have:

$$h_0 \leq \cdots \leq h_k \geq h_{k+1}.$$

Moreover, since $\times\ell : A_k \to A_{k+1}$ is surjective, we conclude $(A/\ell)_{k+1} = 0$. Since A/ℓ is a standard graded algebra, we conclude $(A/\ell)_t = 0$ for every $t \geq k$, which implies $\times\ell : A_i \to A_{i+1}$ is surjective for every $i \geq k$ and so $h_i \geq h_{i+1}$ for every $i \geq k$. $\square$

Proposition 2.5.12 says that from a combinatorial perspective, proving an algebra *satisfies* the WLP has interesting combinatorial consequences. It is well-known that in some settings, failure of the SLP/WLP can be interpreted geometrically, and often represent interesting behaviour. One of the main contributions of this thesis, which will be made clear in later chapters, can be seen as finding interesting combinatorial consequences to an algebra *failing* the WLP.

We end this subsection by stating the following proposition due to Migliore, Miró-Roig and Nagel [148] (for infinite fields), and due to Lundqvist and Nicklasson [125] (for finite fields) that greatly simplifies the process of checking whether an artinian algebra defined by a monomial ideal satisfies the WLP or SLP.

Proposition 2.5.13 ([148, Proposition 2.2], [125, Proposition 4.3])**.** *Let $A = R/I$ be an artinian algebra such that $I \subset R = \mathbb{K}[x_1, \dots, x_n]$ is a monomial ideal. Then A has the WLP (resp. SLP) if and only if $\ell = x_1 + \cdots + x_n$ is a weak (resp. strong) Lefschetz element.*

2.5.3 Gorenstein rings

We now introduce the last class of rings we will need: the class of *Gorenstein rings*. Just as in Sections 2.5.1 and 2.5.2, the motivation we present here is in terms of the conditions in McMullen's g-conjecture. The Cohen-Macaulay property, together with the WLP explain an algebraic process that implies condition 3. in Conjecture 2.4.8. As we will soon see, the Gorenstein property explains 2.: the symmetry of the h-vector.

Before moving on to algebraic notions, it should be noted that in f-vector theory, the corresponding equalities (in terms of the f-vector) that represent symmetry of the h-vector of polytopal spheres were already known since 1927, after Sommerville proved an earlier conjecture from 1905 of Dehn[17]. The equalities are often called *Dehn-Sommerville equations*, and can be stated either in terms of f- or h-vectors:

$$\sum_{j=k}^{d-1} (-1)^{d-j-1} \binom{j+1}{k+1} f_j = f_k \quad \text{or} \quad h_k = h_{d-k}.$$

On the algebra side, the symmetry $h_k = h_{d-k}$ of the h-vector of a *Gorenstein* ring is a consequence that follows almost trivially from several of its (equivalent) definitions. As is pointed out in Bass' paper [9] and by Huneke [98], Gorenstein rings are ubiquitous in mathematics, having appeared in several different contexts, including the study of plane curves, injective resolutions and algebraic topology (Poincaré duality). Here we give a definition that is often not viewed as the standard definition in commutative algebra, but plays a very important role in other areas of mathematics such as topology.

Let $\varphi : V \times W \to \mathbb{K}$ be a bilinear map of $\mathbb{K}$-vector spaces. Note that φ induces linear maps $\varphi_1 : V \to \mathrm{Hom}(W, \mathbb{K})$ and $\varphi_2 : W \to \mathrm{Hom}(V, \mathbb{K})$, where $\mathrm{Hom}(X, \mathbb{K})$ denotes the dual vector space: $\mathrm{Hom}(X, \mathbb{K}) = \{\psi : X \to \mathbb{K} : \quad \psi \text{ is linear}\}$. The bilinear map φ is called a *perfect pairing* if the induced maps φ_1 and φ_2 are isomorphisms.

[17] We refer the reader to [75, Section 9.8] for more historical details and references.

Definition 2.5.14 (**Gorenstein rings and Poincaré duality**, [78, Theorem 2.79])**.** An artinian algebra A of socle degree d is a *Gorenstein* algebra if $\dim \operatorname{soc}(A) = 1$. Equivalently, an artinian algebra is Gorenstein if it satisfies *Poincaré duality*, that is, $A_d \cong \mathbb{K}$ and the pairing induced by multiplication:

$$A_k \times A_{d-k} \to A_d \quad \text{where} \quad (a, b) \mapsto ab$$

is a perfect pairing.

Note that so far we have only defined the Gorenstein property for artinian algebras. We will give more general definitions in later sections. For now, a nonartinian Gorenstein ring can be taken to be a Cohen-Macaulay ring A such that $A/\mathbf{x}$ is Gorenstein, where $\mathbf{x}$ is a (equivalently, any) s.o.p. of A (see Proposition 2.6.10).

The symmetry of the h-vector of a Gorenstein algebra follows directly from its definition.

Proposition 2.5.15. *Let A be an AG (artinian Gorenstein) $\mathbb{K}$-algebra of socle degree d and let $h_i = \dim A_i$. Then for every $0 \le k \le d$:*

$$h_k = h_{d-k}.$$

Proof. Since for every k there is a perfect pairing $A_k \times A_{d-k} \to A_d$, we know $A_k \cong \operatorname{Hom}(A_{d-k}, \mathbb{K})$. The result follows directly since $\operatorname{Hom}(X, \mathbb{K})$ and X have the same dimension. □

From our perspective, the important consequence of a Stanley-Reisner ring being Gorenstein is that it implies exactly the equalities from the Dehn-Sommerville relations. The same equalities appearing in these two different contexts is explained by the following result that can be found in Stanley's book [187][18]:

Theorem 2.5.16 ([187, Part II Theorem 5.1])**.** *Fix $\mathbb{K}$ a field. Let Δ be a simplicial complex such that every vertex of Δ is contained in at least one minimal nonface of Δ. Then the following are equivalent:*

[18] In [187, pp.65] Stanley mentions that the second item is due to a remark of Björner, and the first item is due to Stanley [182].

1. *Δ is a homology sphere over $\mathbb{K}$;*

2. *Δ is a Cohen-Macaulay orientable (over $\mathbb{K}$) pseudomanifold without boundary*

3. *R/I_Δ is Gorenstein, where the base field of R is $\mathbb{K}$.*

We are now ready to finish the characterization of h-vectors of 2-dimensional homology spheres originally due to Steinitz, which we started in Example 2.4.5.

Example 2.5.17. Let Δ be a 2-dimensional simplicial sphere with f-vector $f(\Delta) = (1, v, e, f)$. In Example 2.4.5 we saw that the h-vector of Δ is given by $h(\Delta) = (1, v - 3, e - 2v + 3, f - e + v - 1)$. The Dehn-Sommerville relations imply

$$\begin{cases} v - 3 = e - 2v + 3 \\ 1 = f - e + v - 1. \end{cases}$$

In particular, we conclude $e = 3v - 6$ and $f = 2v - 4$.

In terms of independence complexes of graphs, the following characterization was shown by Hoang and Trung [88] in 2016:

Theorem 2.5.18 ([88, Theorem 3.4])**.** *Let G be a triangle-free graph without isolated vertices. Then the following are equivalent:*

1. $\mathrm{Ind}(G)$ *is a homology sphere;*

2. *G is 1-well-covered (or equivalently, G is a W_2 graph).*

Note that being 1-well-covered is a property that is not affected by the characteristic of any field, so Theorem 2.5.18 in particular says if a triangle-free graph is Gorenstein over some field $\mathbb{K}$, it must be Gorenstein over every field $\mathbb{K}$.

Before moving on to the next section, we define another class of rings which is contained in the class of Gorenstein rings. An ideal $I \subset R = \mathbb{K}[x_1, \dots, x_n]$ is called a *complete intersection* if it is generated by a regular sequence. A monomial ideal is a complete intersection if and only if every generator has disjoint support.

Example 2.5.19. The ideal $I_n = (x_1y_1, \dots, x_ny_n) \subset R = \mathbb{K}[x_1, \dots, x_n, y_1, \dots, y_n]$ is a monomial complete intersection. Note that I_n is the Stanley-Reisner ideal of the complex $\mathrm{Ind}(G)$ where $G = w(H_n)$ and H_n is the graph consisting of n isolated vertices. In particular, the complexes $\mathrm{Ind}(w(H_n))$ are of the form $B(\Delta)$, but since I_n is a complete intersection, I_n is Gorenstein and hence $\mathrm{Ind}(w(H_n))$ is a homology sphere. This shows that $\mathrm{Ind}(w(H_n))$ are examples of the degenerate cases of Bier balls that are actually spheres, mentioned before in Item 2 at the end of Section 2.2.

A proof that these are all the degenerate cases can be found for example in [35].

2.6 Minimal Free Resolutions and Hochster's formula

We now shift gears a little bit to introduce more invariants of squarefree monomial ideals called *(multigraded) Betti numbers*. These invariants can be used to obtain the Hilbert series of an ideal, and hence can be seen as finer invariants than the h-vector.

We note moreover that the first "combinatorial approach" to studying free resolutions of monomial ideals can be found in Taylor's thesis [193] in 1966, where she introduced a famous (nonminimal combinatorial) resolution of monomial ideals. Similarly to Stanley-Reisner theory, Taylor's thesis could also be pinpointed as the beginning of combinatorial commutative algebra.

Let $I = (f_1, \ldots, f_s)$ be a homogeneous ideal of $R = \mathbb{K}[x_1, \ldots, x_n]$. There is a map

$$d_0 : F_1 \to F_0 = R, \quad d_0(\mathbf{e}_i) = f_i$$

where the $\mathbf{e}_i$ form the canonical basis of the free module F_1. The map d_0 has a finitely generated kernel $K \subset F_1$, hence there is a map

$$d_1 : F_2 \to F_1, \quad d_1(\mathbf{e}'_j) = g_j$$

where g_j are the generators of K and the $\mathbf{e}'_j$ form the canonical basis of the free module F_2. Repeating this process, we have an exact chain complex of free modules which is called the *Minimal Free Resolution* (MFR) of R/I:

$$0 \to F_p \longrightarrow \ldots F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow R/I \to 0,$$

where the fact that this procedure in fact stops after a finite number of steps is the following well-known result due to Hilbert.

Theorem 2.6.1 (Hilbert syzygy theorem). *The free resolution of any homogeneous ideal in $R = \mathbb{K}[x_1, \dots, x_n]$ ends after at most n steps.*

Proof. See [159, Theorem 15.2]. □

The following result allows us define several invariants of an ideal associated to minimal free resolutions.

Proposition 2.6.2. *If C and C' are two minimal free resolutions of a $\mathbb{N}^s$-graded ideal I of $R = \mathbb{K}[x_1, \dots, x_n]$, then:*

$$\dim_{\mathbb{K}}(F_i)_{\mathbf{a}} = \dim_{\mathbb{K}}(F_i')_{\mathbf{a}} \quad \textit{for all} \quad \mathbf{a} \in \mathbb{N}^s,$$

where F_i and F_i' are the i-th free modules in C and C' respectively.

Proof. See [159, Chapter 26]. □

Definition 2.6.3. The length p of a minimal free resolution of R/I is called the *projective dimension* of R/I, denoted by $\operatorname{pdim}(R/I)$. The numbers $\beta_i(R/I) = \operatorname{rank} F_i$ are called the *Betti numbers* of R/I.

Every free module F_i in a minimal free resolution of R/I can be written as

$$F_i = \bigoplus_j R(-j)^{\beta_{ij}}$$

where $R(-j)$ is a free module of rank 1 such that the element 1 has degree j. The numbers $\beta_{ij}(R/I)$ are called the *graded Betti numbers* of R/I.

When I is a monomial ideal, not only is R/I an $\mathbb{N}$-graded R-module, it is also an $\mathbb{N}^n$-graded R-module. In particular, the free modules F_i in a minimal free resolution of R/I have a decomposition as

$$F_i = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} R(-\mathbf{a})^{\beta_{i,\mathbf{a}}(R/I)},$$

where $R(-\mathbf{a})$ is the free R-module of rank 1 such that the element 1 has degree $\mathbf{a}$. The numbers $\beta_{i,\mathbf{a}}(R/I)$ are called the *multigraded Betti numbers* of R/I. Since we can take the entries of $\mathbf{a} = (a_1, \dots, a_n)$ to be exponents of the monomial $m = x_1^{a_1} \dots x_n^{a_n}$, the Betti number $\beta_{i,\mathbf{a}}(R/I)$ is also written as $\beta_{i,m}(R/I)$.

The *Castelnuovo-Mumford regularity* (or simply regularity) of S/I is

$$\operatorname{reg}(R/I) = \max(j - i \colon \beta_{i,j}(R/I) \neq 0).$$

The projective dimension is directly related to the notion of depth introduced in Section 2.5.

Theorem 2.6.4 (**Auslander-Buchsbaum formula,** [27, Chapter 1])**.** *Let* $R = \mathbb{K}[x_1, \dots, x_n]$ *and* I *an ideal of* R*. Then*

$$n = \operatorname{pdim}(R/I) + \operatorname{depth}(R/I).$$

The following proposition, which is a general version of the well-known rank-nullity theorem for linear algebra, allows us to relate free resolutions to Hilbert functions.

Proposition 2.6.5. *Let*

$$0 \to F_p \longrightarrow F_{p-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \to R/I \to 0$$

be a long exact sequence of free R*-modules. Then for every* i

$$\operatorname{HF}_{R/I}(i) = \sum_{j=0}^{p} (-1)^j \operatorname{HF}_{F_j}(i).$$

In particular, we have

$$\operatorname{HS}_{R/I}(t) = \sum_{j=0}^{p} (-1)^j \operatorname{HS}_{R/I}(t).$$

Proof. The statement is equivalent to saying the alternate sum of dimensions in an exact sequence of vector spaces is equal to 0. The result then follows from [173, Exercise 3.16]. □

Although the definitions of free resolutions and Betti numbers are very algebraic in nature, they turn out to have very natural combinatorial interpretations for monomial ideals. Computing the Hilbert series of a monomial ideal is essentially equivalent to counting the number of ways we can put i apples in n boxes following some restrictions given by the monomial ideal. The free resolution of R/I should be thought of as the abstract generalization of the inclusion-exclusion method for solving this problem, where not only do we have an alternate sum for the dimensions, but we also have a long exact sequence that gives an algebraic explanation for the alternating sum. The next example is a concrete comparison between the free resolution of a small monomial ideal, and the corresponding apples and boxes problem.

Example 2.6.6 (An "apples and boxes" approach to free resolutions[19]). Let

$$I = (x_1x_2, x_3x_4) \subset \mathbb{K}[x_1, \ldots, x_4].$$

Note that I is the Stanley-Reisner ideal of a square (a 1-dimensional homology sphere), and hence is Gorenstein by Theorem 2.5.16. Computing the minimal free resolution of R/I using Macaulay2 [73] we have:

$$0 \to R(-4) \longrightarrow R(-2)^2 \longrightarrow R \to R/I \to 0.$$

The Hilbert series of R/I is then given by

$$\begin{aligned} \mathrm{HS}_{R/I}(t) &= \mathrm{HS}_R(t) - 2t^2\,\mathrm{HS}_R(t) + t^4\,\mathrm{HS}_R(t) && (2.2)\\ &= \frac{1}{(1-t)^4} - \frac{2t^2}{(1-t)^4} + \frac{t^4}{(1-t)^4} && (2.3)\\ &= \frac{1-2t^2+t^4}{(1-t)^4} = \frac{(1-t)^2(1+t)^2}{(1-t)^4} = \frac{t^2+2t+1}{(1-t)^4}. && (2.4)\end{aligned}$$

Now note that computing the Hilbert function of R/I at t is equivalent to computing in how many ways can we put t undistinguishable apples in 4 distinct boxes, such that if box 1 (respectively 3) has an apple, then box 2 (respectively 4) must be empty.

We can solve this problem by applying the inclusion-exclusion principle: we first consider the number of ways we can put t apples in 4 boxes without any restriction, which is equal to $\mathrm{HF}_R(t)$ by a standard *stars and bars* argument. Then, for every generator x_ix_j of I, we need to subtract the number of ways we can put t apples inside the 4 boxes, assuming box i and j both have at least one apple inside them. Again by a stars and bars argument

[19] The procedure described here, when generalized, would correspond to the (often nonminimal) resolution introduced by Taylor in her thesis [193] in 1966.

this is equal to $\mathrm{HF}_R(t-2)$ for every generator. Now, we have to add back the arrangements where we broke 2 rules (one coming from x_1x_2 and another coming from x_3x_4), in other words, where we assume boxes $1, 2, 3, 4$ all have at least one box inside them. This corresponds to the "intersection" of the cases corresponding to a single edge, and amount to $\mathrm{HF}_R(t-4)$. Adding up the terms we have

$$\mathrm{HF}_{R/I}(t) = \mathrm{HF}_R(t) - 2\,\mathrm{HF}_R(t-2) + \mathrm{HF}_R(t-4).$$

Rewriting the equation above in terms of Hilbert series we get (2.2).

There are many factors contributing to the simple computation of Example 2.6.6. One of them is that the resolution is small and there is no cancellation of any terms. If we were to try the same process with the Stanley-Reisner ideal of a pentagon, even though in principle the same process would work, it would be unclear why some terms cancel and do not appear in the final formula.

It turns out that in order to understand the "minimal decomposition" of the solution to the stars and bars problem with restrictions arising from I_Δ, in terms of stars and bars problems without restrictions (i.e. computing a minimal free resolution), one has to understand the homology of induced subcomplexes of Δ. This result is known as *Hochster's formula* and was originally shown by Hochster [89] in 1977 in a slightly different setting.

Theorem 2.6.7 (Hochster's formula, [89]**).** *Let $R = \mathbb{K}[x_1, \ldots, x_n]$ and Δ a simplicial complex on vertex set $[n]$. Then*

$$\beta_{i,m}(R/I_\Delta) = \dim_{\mathbb{K}} \tilde{H}_{\deg m - i - 1}(\Delta_{\mathrm{supp}(m)}; \mathbb{K}).$$

In Example 2.6.6 for example the nonzero multidegrees in a minimal free resolution of I are: $1, x_1x_2, x_3x_4$ and $x_1x_2x_3x_4$. These correspond to the induced subcomplexes of a square that have nontrivial homology, where the vertices inducing the complexes are: $\emptyset, \{1, 2\}, \{3, 4\}$ and $\{1, 2, 3, 4\}$.

The sequence of Betti numbers $\beta(\Delta) = (\beta_0(R/I_\Delta), \beta_1(R/I_\Delta), \beta_2(R/I_\Delta)) = (1, 2, 1)$ from Example 2.6.6 is palindromic, just as the h-vector of a homology sphere. This is not a coincidence, and is in fact another consequence of the Gorenstein property: the resolution of a Gorenstein ring is *self-dual*. Throughout this text we will not need the full strength of this result, but we will need the notion of "canonical modules", which are objects that play key roles in the theory of Gorenstein rings.

The standard definition of canonical module requires several prerequisites that we have not introduced. Here, we will follow Stanley's [187, I Chapter 12] approach to defining the canonical module for a quotient of the polynomial ring.

Before defining it, note that given an exact complex C, we may take the "transpose" of C by replacing every module F of C by the dual

$$\operatorname{Hom}(F, R) := \{\psi : F \to R \colon \psi \text{ is an } R\text{-module homomorphism}\},$$

and the maps become the transposes of the maps in C. This new complex, which we denote by $\operatorname{Hom}(C, R)$ is still a complex, since composing two consecutive maps still gives the 0 map, but it need not be exact[20].

Definition 2.6.8 (Canonical module for quotients of the polynomial ring). Let $R = \mathbb{K}[x_1, \dots, x_n]$, $I \subset R$ a homogeneous ideal such that R/I is a d-dimensional Cohen-Macaulay ring and let

$$C := \quad 0 \to F_p \xrightarrow{d_p} F_{p-1} \xrightarrow{d_{p-1}} \dots \xrightarrow{d_1} F_1 \xrightarrow{d_0} R \to R/I \to 0,$$

be the minimal free resolution of R/I. The *canonical module* of R/I is the quotient

$$\omega_{R/I} = \frac{\operatorname{Hom}(F_p, R)}{\operatorname{im} d_p^\top},$$

in other words, the last homology of $\operatorname{Hom}(C, R)$.

We are now ready to give a second definition of Gorenstein rings. This definition also works for nonartinian rings.

Theorem 2.6.9. *Let $R = \mathbb{K}[x_1, \dots, x_n]$ and $I \subset R$ a homogeneous ideal. The following are equivalent:*

[20] This operation can be more generally introduced as applying the functor $\operatorname{Hom}(-, R)$ to the complex C. If C is a minimal free resolution of an R-module M, the homology of the new complex is (by definition) the derived functor of $\operatorname{Hom}(-, R)$, which is usually denoted by $\operatorname{Ext}^i_R(M, R)$. We refer the reader to [173] for the required background for the more general terminology in homological algebra, and to [27] for the general results relevant to commutative algebra.

1. *R/I is a Gorenstein ring*

2. *R/I is a d-dimensional Cohen-Macaulay ring and $\beta_{n-d}(R/I) = 1$.*

3. *$\omega_{R/I} \cong R/I(-a)$ for some integer a and R/I is Cohen-Macaulay.*

Proof. See [27, Theorem 3.2.10] for the equivalence $1. \iff 2.$ and [27, Proposition 3.6.11] for the equivalence $1. \iff 3.$ □

We note that even though the definition of canonical module is abstract, it is often the case that the canonical module of rings arising in combinatorics have very natural (and combinatorial) descriptions, see [187, I Chapter 13] for an example of this phenomenon.

In Section 2.5 we mentioned that we could take a nonartinian Gorenstein ring to be a Cohen-Macaulay ring A such that $A/\mathbf{x}$ is Gorenstein, where $\mathbf{x}$ is any s.o.p. of A. The reasons both definitions are equivalent are gathered in the following proposition.

Proposition 2.6.10. *Let $A = R/I$ be a standard graded Gorenstein ring. Then:*

1. *if x is an A-regular element, then A/x is Gorenstein.*

2. *if A is artinian, then $\dim \operatorname{soc}(A) = \beta_n(A)$ where $n = \dim R$. In particular, both definitions of artinian Gorenstein rings are equivalent.*

Proof. 1. See [27, 3.1.19(b)].

2. For the equality, see [159, Corollary 14.12]. The equivalence of conditions then follows since artinian rings are 0 dimensional and hence CM ($0 \leq \operatorname{depth} A \leq \dim A = 0$ by Proposition 2.5.3), and by Theorem 2.6.9.

□

Finally, there is a convenient way of visualizing the relevant information in a resolution. The *Betti table* of R/I is a matrix $\beta(R/I)$, often written down as a table, such that the (i, j)–th entry of $\beta(R/I)$ is the Betti number $\beta_{i,i+j}(R/I)$. Several invariants can be read off from the Betti table, below we list the ones we introduced in this section:

1. The projective dimension of R/I is the number of columns of $\beta(R/I)$;

2. The regularity of R/I is the number of (nonzero) rows of $\beta(R/I)$.

	0	1	2	3	4	5	6
total:	1	26	99	167	147	66	12
0 :	1	.	.	.	.	.	.
1 :	.	9	15	9	2	.	.
2 :	.	17	84	158	145	66	12

	0	1	2	3	4	5	6	7
total:	1	26	99	167	147	66	13	1
0 :	1	.	.	.	.	.	.	.
1 :	.	9	15	9	2	.	.	.
2 :	.	17	84	158	145	66	12	1
3 :	.	.	.	.	.	.	1	.

Figure 2.4: The Betti tables of R/I_Δ, where $I_\Delta \subset R = \mathbb{K}[x_1, \ldots, x_9]$, Δ is the triangulation of the Moore space from Example 2.1.3, and $\mathbb{K} = \mathbb{Q}$ (left) and $\mathbb{K} = \mathbb{Z}_3$ (right).

Remark 2.6.11 (Torsion and resolutions). By Theorem 2.6.7, the minimal free resolution of R/I can change based on the underlying field. If for example $I = I_\Delta$ is the Stanley-Reisner ideal of a simplicial complex Δ that contains an induced complex with torsion, then the torsion will change the Betti numbers depending on the characteristic of the underlying field. In Figure 2.4 we see that the Stanley-Reisner ideal I_Δ of the Moore space from Example 2.1.3 has different Betti tables based on the field. If the field has characteristic 3 the regularity is 3, while the regularity is only 2 if the characteristic is not 3. Even the Cohen-Macaulay property ends up depending on the field: the ring R/I_Δ from Figure 2.4 is CM if and only if the characteristic of the underlying field $\mathbb{K}$ is not 3. Moreover, for a squarefree monomial ideal I, there are finitely many characteristics that can affect the resolution, and because of Moore spaces introduced in Example 2.1.3, for any subset of primes A, there exists a squarefree monomial ideal with a different resolution exactly for fields with characteristic in A.

Finally we note that even though the Betti table in Figure 2.4 changed in characteristic 3, the changes appear in the same "anti-diagonal" of the Betti table. This is not a coincidence, as it can be shown that Betti numbers in such anti-diagonals cancel each other and do not affect the Hilbert series. Since the Hilbert series of R/I_Δ is not affected by the characteristic, the changes in the Betti table must always happen in anti-diagonals. From a more topological perspective, the different entries appearing in the same anti-diagonals can also be seen from the *universal coefficient theorem* (see [80, 3.A]).

2.7 Edge (facet) ideals and Rees algebras

In this section we introduce the second construction to associate ideals to graphs, and more generally, simplicial complexes[21].

Definition 2.7.1. Let $\Delta = \langle F_1, \ldots, F_s \rangle$ be a simplicial complex on vertex set V. The *facet ideal* of Δ is the ideal

$$I(\Delta) = (x_{F_1}, \ldots, x_{F_s}) \subset \mathbb{K}[x_v \colon v \in V] \quad \text{where} \quad x_{F_i} = \prod_{j \in F_i} x_j.$$

When Δ is a graph, we call $I(\Delta)$ the *edge ideal* of Δ.

Facet ideals were first introduced by Villarreal [197] in 1990 for graphs, where he first used the term edge ideals. Later in 2002, Faridi [53] generalized the construction to arbitrary simplicial complexes.

While Stanley-Reisner theory has proven to be intrinsically related to more topological questions about simplicial complexes, facet ideal theory turns out to be related to questions arising in combinatorial optimization. Just as many open conjectures have turned out to be equivalent to questions in Stanley-Reisner theory, the same phenomenon happens with conjectures in combinatorial optimization and questions in facet ideal theory. The most famous example of this relationship is the connection between the *Conforti-Cornuéjols* conjecture, and the study of associated primes of powers of facet ideals. We will not need any concept from combinatorial optimization throughout this thesis, and we refer the reader to [150] for more details on the applications of facet ideals (there called edge ideals) to combinatorial optimization.

Definition 2.7.2 (Rees algebras). Let I be an ideal of $R = \mathbb{K}[x_1, \ldots, x_n]$. The *Rees ring* of I is the ring

$$\mathcal{R}(I) := \bigoplus_{n \in \mathbb{N}} I^n,$$

where by convention $I^0 = R$.

[21] In the literature, hypergraphs are often used instead of simplicial complexes. Since we already need simplicial complexes for Stanley-Reisner theory, we use simplicial complexes instead. Note that both results from hypergraph theory, as well as results from simplicial complexes have been shown to be useful for different types of problems.

The *fiber ring* of I is the ring

$$\mathcal{F}(I) = \frac{\mathcal{R}(I)}{\mathfrak{m}\mathcal{R}(I)} = \bigoplus_{n \in \mathbb{N}} \frac{I^n}{\mathfrak{m}I^n} \quad \text{where} \quad \mathfrak{m} = (x_1, \ldots, x_n).$$

The *analytic spread* of I is the dimension $\ell(I) = \dim \mathcal{F}(I)$.

The Rees algebra was originally defined by David Rees [169] in 1958, where he was considering a problem due to Zariski related to Hilbert's 14th problem. Intuitively, the Rees algebra of I contains information about all the powers of I.

In order to clearly state the connection between Rees algebras and Lefschetz properties, we first need the following result.

Given two fields K, L such that $K \subseteq L$, a *transcendence basis* of L over K is a subset T of L such that L is an algebraic field extension of $K(T)$. The *transcendence degree* of L over K, denoted by $\operatorname{trdeg}_K(L)$ is the size of a transcendence basis of L over K. If A is a domain, denote by $Q(A)$ the fraction field of A.

Proposition 2.7.3. *Let $\mathbb{K}$ be a field and A an integral domain which is finitely generated over $\mathbb{K}$. Then*

$$\dim A = \operatorname{trdeg}_{\mathbb{K}} Q(A),$$

In particular, if $I = (f_1, \ldots, f_s) \subset R = \mathbb{K}[x_1, \ldots, x_n]$ is a homogeneous ideal generated by polynomials of the same degree, then

$$\ell(I) = \operatorname{trdeg}_{\mathbb{K}}(\mathbb{K}(f_1, \ldots, f_s)).$$

Proof. The first part of the statement can be found in [135, Theorem 5.6]. The second part of the statement follows from [83, Proposition 4.8], but we include a proof for the sake of completeness.

Let $\mathfrak{m} = (x_1, \ldots, x_n)$. Since $I = (f_1, \ldots, f_s)$ where $\deg f_i = d$ for all i, we have

$$\frac{I^k}{\mathfrak{m}I^k} \cong I^k \cap R_{dk} \quad \text{for all} \quad k \geq 0.$$

We then conclude

$$\mathcal{F}(I) \cong \mathbb{K}[f_1, \ldots, f_s],$$

which means $\mathcal{F}(I)$ is a domain and so by the first part we have

$$\ell(I) = \dim \mathcal{F}(I) = \operatorname{trdeg}_{\mathbb{K}}(Q(\mathcal{F}(I))) = \operatorname{trdeg}_{\mathbb{K}}(\mathbb{K}(f_1, \dots, f_s)).$$

□

Computing the transcendence of a field extension of the form $\mathbb{K} \subseteq \mathbb{K}(f_1, \dots, f_s)$ is a common problem in many areas of mathematics. One of the first times (if not the first) where this question was considered, was in the context of *algebraic matroids*, which are a subclass of *matroids*: objects that generalize the notion of linear independence. From this perspective, the notion of transcendence degree is exactly the same as the notion of dimension of a set of vectors. The computation (in characteristic 0 and in large enough characteristic) of the transcendence degree is often done using the *Jacobian criterion*, which says a set of polynomials is algebraically independent if and only if the corresponding Jacobian matrix has full rank. See [44, 47, 177] for proofs of these statements from different perspectives. Note that the entries of the Jacobian matrix are by definition polynomials. In order to compute the rank of such matrices, in the literature there are several approaches for example in terms of "randomized evaluations", where a point is chosen at random (under some suited distribution), and the rank of the Jacobian matrix is equal to the rank of the evaluated matrix with high probability. An algebraic explanation as to why such approaches work follows from the fact that every open set in the Zariski topology is dense.

For monomial ideals, the situation is simpler. There is a simple combinatorially defined matrix that allows us to compute the analytic spread of a monomial ideal generated in a single degree.

Definition 2.7.4 (Log-matrix). Let $I = (m_1, \dots, m_s) \subset S = \mathbb{K}[x_1, \dots, x_n]$ be a monomial ideal. The *log-matrix* of I, denoted by $\log(I)$ is the matrix such that its rows are the exponent vectors of the monomials $m_1, \dots, m_s$. If the sum of every row of a matrix N is equal to some fixed number d, we say N is a *d-stochastic* matrix. In particular, if the generators of I have the same degree d, the log-matrix of I is d-stochastic.

Theorem 2.7.5. *The analytic spread of a monomial ideal I generated by monomials of degree d is equal to* $\operatorname{rank} \log(I)$.

Proof. See [198, Proposition 7.1.17 and Exercise 7.4.10]. □

In later chapters we will show that matrices representing multiplication maps in monomial algebras are often equal to the log-matrix of some equigenerated monomial ideal.

Chapter 3

Failing "for a reason": unexpected systems of parameters

"I may not triumph in success,
Despite my earnest labor;
I may not grasp results that bless
The efforts of my neighbor;
But though my goal I never see
This thought shall always dwell with me.

The golden glory of Love's light
May never fall on my way;
My path may always lead through night,
Like some deserted by-way;
But though life's dearest joy I miss
There lies a nameless strength in this."

ELLA WHEELER WILCOX – *"I'll be worthy of it"*

In Section 2.5 we motivated the study of Lefschetz properties by relating the existence of rings *satisfying* the weak Lefschetz properties (WLP) to the classification of f-vectors of homology spheres. The aim of this chapter is to provide a deeper study of the *failure* of the WLP for (squarefree) monomial ideals. We note that in most settings, the WLP is expected to hold, and failure is often seen as an anomaly. However, in (algebraic and differential) geometry, the failure of the WLP often leads to interesting geometric phenomena. In this chapter, we try to adapt this framework to a more combinatorial setting.

The main inspiration for most of our results in this chapter is the theory of unexpected hypersurfaces due to Cook, Harbourne, Migliore and Nagel [32]. We note that a combinatorial motivation for studying inverse systems was already developed by Lee [116] in view of the connections with rigidity theory, which we recall in Section 3.2. This chapter is based on [93, 94].

3.1 An algebraic theory: inverse systems

In 1916, Macaulay [128] introduced his theory of *inverse systems*, which we now recall.

Let $R = \mathbb{K}[x_1, \ldots, x_n]$ and $S = \mathbb{K}[y_1, \ldots, y_n]$ be two polynomial rings such that $\deg x_i = 1$ and $\deg y_i = -1$. We may view S as an R-module via *contraction*, which is defined on monomials by:

$$x_1^{a_1} \ldots x_n^{a_n} \circ y_1^{b_1} \ldots y_n^{b_n} = \begin{cases} y_1^{b_1 - a_1} \ldots y_n^{b_n - a_n} & \text{if} \quad a_i \leq b_i \quad \text{for all} \quad i \\ 0 \quad \text{otherwise} \end{cases} \tag{3.1}$$

and is easily extended (linearly) to arbitrary polynomials in R and S. Note that since $\deg y_i = -\deg x_i = -1$, S is a graded R-module.

Given an ideal $I \subset R$, the *inverse system* of I is the (graded) S-module:

$$I^{-1} = \{g \in S \colon f \circ g = 0 \quad \text{for all} \quad f \in I\}.$$

Example 3.1.1 (Inverse systems of artinian monomial complete intersections). Let $I = (x_1^{c_1}, \ldots, x_n^{c_n}) \subset R$. Then I^{-1} consists of polynomials f such that every variable y_i appears with power at most $c_i - 1$ in f. Since every such polynomial is a sum of monomials obtained after repeatedly acting by x_i on $y_1^{c_1 - 1} \ldots y_n^{c_n - 1}$ we have:

$$I^{-1} = (y_1^{c_1 - 1} \ldots y_n^{c_n - 1}) \subset S.$$

More generally, the inverse system of an artinian monomial ideal J consists of the monomials not in J (after replacing every x_i by y_i).

In Example 3.1.1 we see that the inverse system of an artinian monomial complete intersection is:

1. cyclic (has a single generator)

2. finite (by finite we mean that I^{-1} is a finite dimensional $\mathbb{K}$-vector space)

Moreover, we also have that $\dim(R/I)_i = \dim(I^{-1})_{-i}$ for every $i \geq 0$. None of these is a a coincidence, in fact, they consist of the main properties of inverse systems that we will need.

Theorem 3.1.2 (Macaulay duality). *Let $I \subset R$ be an ideal and $I^{-1} \subset S$ its inverse system. Then*

1. *R/I is artinian if and only if I^{-1} is a finite dimensional $\mathbb{K}$-vector space,*

2. *if $\dim \operatorname{soc}(R/I) = k$, then I^{-1} has k generators. In particular, R/I is artinian Gorenstein if and only if I^{-1} is a finite cyclic R-module, and*

3. *if R/I is artinian, for every i we have $\dim(R/I)_i = \dim I^{-1}_{-i}$. In particular, the socle degree of R/I is equal to the highest absolute value of the degrees of generators of I^{-1}.*

Proof. See [78, Section 2.4.1 and 2.4.2]. □

When R/I is an artinian Gorenstein algebra, we call the polynomial F such that $I^{-1} = (F)$ the *Macaulay dual generator of* R/I. To every finite R-module $W \subset S$, there exists an ideal $I \subset R$ such that R/I is artinian and $I^{-1} = W$. In other words, Macaulay duality says there is a bijection (up to multiplying by constants) between artinian Gorenstein algebras of the form R/I of socle degree d, and homogeneous polynomials $F \in S$ of degree d.

Remark 3.1.3 (Differentiation and contraction). Just as we viewed S as an R-module by contraction, if the characteristic of the underlying field is 0, we may also view S as an R-module via *differentiation*:

$$x_i \bullet g = \frac{\partial g}{\partial y_i}.$$

In this setting, we denote the inverse system of I by $I^\top$, so there is no confusion as to which action we are considering. Every part of Theorem 3.1.2 still holds. It is not true, however, that if $(F) = I^{-1} = J^\top$ for two ideals $I, J \subset R$, that we have $R/I \cong R/J$. As an example, let $F = (y_1 - y_2)(y_2 - y_3)(y_3 - y_4) \in \mathbb{Q}[y_1, \ldots, y_4]$. Then the ideal I such that $I^{-1} = (F)$ has 6 generators, while the ideal J such that $J^\perp = (F)$ has 4 generators.

Remark 3.1.4 (A slight (very helpful) abuse of notation). By a slight abuse of notation, we may treat the elements of $I^{-1} \subset S$ as elements of R by simply replacing every y_i by x_i. We will frequently do this from now on, and in fact this simple change is the key insight in the main results of this chapter (and Chapter 4).

Remark 3.1.5 (Matlis duality). Macaulay duality can be seen as the 0-dimensional case of *Matlis duality*, introduced by Matlis [132] in 1958. The "duality" in the theory introduced by Matlis comes from the fact that for certain (local and complete, with maximal ideal $\mathfrak{m}$) rings, there is an isomorphism $M \cong \mathrm{Hom}(\mathrm{Hom}(M, E), E)$, where M is a finitely generated (or artinian) module and E is a special module called the *injective envelope* of $R/\mathfrak{m}$. Although this is stated in a higher generality than what we need, it should be noted that in our case, E is the polynomial ring with negative degrees appearing in the definition of inverse systems (see [78, Section 2.4.1]). In particular, we will see that there is a relationship between multiplication maps and contraction maps given by transposing matrices (see Lemma 3.4.3). This phenomenon has in fact already been mentioned indirectly in the definition of canonical module that was given in Section 2.6.

For more details on Matlis duality we refer the reader to [22, Chapter 10].

3.2 A combinatorial motivation: (linear and affine) stresses

In Chapter 2 we introduced McMullen's g-conjecture, which gives a conjectural classification of the f-vectors of homology spheres by saying (in particular) that the g-vector of a homology sphere is an O-sequence. A weaker condition than saying the g-vector is an O-sequence, is saying that the entries of the g-vector are nonnegative, that is,

$$g_i \geq 0 \quad \text{for all} \quad i \geq 0. \tag{3.2}$$

Equation (3.2) is one part[1] of the *Generalized Lower Bound Conjecture* (GLBC), proposed by McMullen and Walkup [139] in 1971 for simplicial polytopes[2].

Even after Stanley's proof of the g-theorem (which implies Equation (3.2) for boundary complexes of simplicial polytopes), new proofs of the same inequalities were still sought

[1]The second part of the GLBC characterizes the cases when $g_i = 0$ for the boundary complex of a simplicial polytope. In view of Stanley's proof of the g-theorem for simplicial polytopes, the second part of the GLBC is the part that remained open until Murai and Nevo [153] proved the GLB theorem in 2013.

[2]The paper [139] ends with the following: "Nevertheless, there are real differences as well as deep theoretical questions to be met with in extending results on simplicial polytopes to triangulated spheres [...] We have therefore satisfied ourselves with venturing the Generalized Lower Bound Conjecture for polytopes only."

after[3]. Roughly 10 years after Stanley's original proof of the g-theorem, in 1987 Kalai [103] gave an alternative proof of the fact that $g_2 \geq 0$ for the boundary complex of any simplicial polytope. Kalai's proof used techniques from *rigidity theory*, more specifically, the study of stresses and flexes of graphs embedded in euclidean spaces. We now introduce the required background in rigidity theory for our purposes in this chapter and in Chapter 4.

A *bar-and-joint framework* in $\mathbb{R}^d$ is a tuple (G, p) where G is a graph on vertex set $V = [n]$ and $p : V \to \mathbb{R}^d$ is an embedding of the vertices of G in $\mathbb{R}^d$, since p embeds G in $\mathbb{R}^d$ we say p is an d-embedding. Given a graph G on vertex set $V = [n]$, an d-embedding $p : V \to \mathbb{R}^d$ of G is called *d-rigid* if any small perturbation p' of p that preserves edge lengths of G extends to an isometry of $\mathbb{R}^d$. In other words, an d-embedding p of G is d-rigid if there exists $\varepsilon > 0$ such that for every d-embedding p', if $|p'_i - p_i| < \varepsilon$ for every i and p' satisfies

$$||p'_i - p'_j||^2 = ||p_i - p_j||^2 \quad \text{for all} \quad ij \in G,$$

then there exists an isometry $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ such that $\varphi(p_i) = p'_i$ for all i.[4] A graph G is *generically d-rigid* if the space of all d-rigid embeddings of G is an open dense set in the space of all d-embeddings (which is a subspace of the space of functions from V to $\mathbb{R}^d$)

An *infinitesimal flex* of a d-embedding p of G is another d-embedding p' such that

$$(p_i - p_j)^\top (p'_i - p'_j) = 0 \quad \text{for all} \quad ij \in G.$$

If an infinitesimal flex p' of p satisfies $(p_i - p_j)^\top (p'_i - p'_j) = 0$ for every pair of vertices i, j of G, then we say p' is *trivial*. A d-embedding p of G is *infinitesimally rigid* if every infinitesimal flex of G is trivial.

It turns out that if there exists an d-embedding p of G such that p is infinitesimally rigid, then G is generically d-rigid (see for example [103, Section 3]).

A *stress* on a bar-and-joint framework (G, p) is an assignment of numbers λ_{ij} to the

[3]In 1993, McMullen [140] says: "One main aim has therefore been to find a proof of this necessity which avoids such heavy machinery, preferably entirely within convexity." In 2014, during Stanley's 70-th birthday conference, Billera [12] mentions in one of his slides: "Or, as he [McMullen] once (only half-jokingly) put it, '*ridding the subject of this malignancy*'"

[4]Here we tried to give the most direct definitions for our purposes. For more complete definitions the reader should check for example Kalai's paper [103, Section 3], where he gives more abstract definitions in terms of isometries, metric spaces and distances. In particular, an embedding p of a graph G is said to be rigid if there exists an $\varepsilon > 0$ such that every embedding p' of G that is G-isometric to p and the distance between p and p' is bounded above by ε, it must be that p' and p are isometric.

edges $ij \in G$ such that

$$\sum_{j\colon ij \in G} \lambda_{ij}(p_i - p_j) = 0 \quad \text{for all} \quad j \in V. \tag{3.3}$$

The set of all stresses is called the *stress space* of (G, p).

The key insight in Kalai's [103] proof that $g_2 \geq 0$ for every boundary complex of simplicial polytopes was that $g_2(P)$ is equal to the dimension of the space of stresses of the graph of the simplicial polytope P, with respect to a generic embedding.

In 1996, Lee [116] mentions[5] that Kalai speculates whether it is possible to generalize Kalai's techniques from rigidity theory in order to obtain new proofs of the generalized lower bound inequalities for simplicial polytopes. In the same paper, Lee begins the study of rigidity theory from the perspective of the theory of inverse systems due to Macaulay, which we recalled in Section 3.1.

Lee's idea can be explained via Macaulay duality from the following equalities, which follow from the g-theorem due to Stanley [185], and Kalai's proof of $g_2 \geq 0$ [103]:

$$\dim\left(\frac{R}{I_\Delta + (\theta_1, \ldots, \theta_{d+1}, L)}\right)_2 = g_2 = \text{ dimension of the space of stresses of the 1-faces of } \Delta$$

and by Macaulay duality we may write the first term in the equality above as the dimension of the (-2)-graded piece of the inverse system of $I_\Delta + (\theta_1, \ldots, \theta_{d+1}, L)$, where $\theta_1, \ldots, \theta_{d+1}$ is a s.o.p. of I_Δ and L is a Lefschetz element. Moreover, it is known (see for example [187, III Lemma 2.4] and [27, Theorem 5.1.16]) that we may obtain a s.o.p. from either a generic d-embedding of a simplicial $(d-1)$–sphere, or by taking the vertex coordinates of a simplicial polytope: let $p_1, \ldots, p_n \in \mathbb{R}^d$ be either the vertex coordinates of a simplicial polytope with boundary complex Δ, or the images of a generic embedding of a simplicial sphere Δ. Define

$$\theta_j = \sum_{i=1}^{n} (p_i)_j x_i \in R = \mathbb{R}[x_1, \ldots, x_n] \quad \text{for all} \quad j = 1, \ldots, d.$$

Then $\theta_1, \ldots, \theta_d$ form a s.o.p for I_Δ. Knowing this correspondence between geometric objects (embeddings) and algebraic objects (s.o.p.s), Lee defined *affine* and *linear stresses*

[5] Although we could not find a direct reference for this speculation, it is a very natural question considering the generalized lower bound inequalities were already conjectured at that point.

in terms of Macaulay duality.

Definition 3.2.1 (Affine and linear stresses, [116]**).** Let Δ and p be as above. The space of *linear stresses* of Δ with respect to p is:

$$\begin{aligned}\mathcal{S}(\Delta, p) &= \{g \in R \colon f \bullet g = 0 \quad \text{for all} \quad g \in I_\Delta + (\theta_1, \dots, \theta_d)\} &(3.4)\\ &= (I_\Delta + (\theta_1, \dots, \theta_d))^\perp. &(3.5)\end{aligned}$$

Let $L = x_1 + \cdots + x_n$. The space of *affine stresses* of Δ with respect to p is:

$$\begin{aligned}\mathcal{S}^a(\Delta, p) &= \{g \in R \colon f \bullet g = 0 \quad \text{for all} \quad g \in I_\Delta + (\theta_1, \dots, \theta_d, L)\} &(3.6)\\ &= (I_\Delta + (\theta_1, \dots, \theta_d, L))^\perp. &(3.7)\end{aligned}$$

Note that the spaces of affine and linear stresses are both graded since the corresponding inverse system is graded. An (affine or linear) stress of degree i is called an (affine or linear) i-stress of Δ with respect to p.

At first it might seem strange that the correct generalization of stresses is using inverse systems, but the following example should clarify that the "balancing conditions" (3.3) that appear in rigidity theory can be understood from the perspective of differentiating polynomials, which was Lee's idea in [116].

Example 3.2.2 (Affine 2-Stresses and derivatives). Let Δ be a $(d-1)$–dimensional simplicial complex that is either the boundary complex of a simplicial polytope, or a simplicial sphere. Let p be an d-embedding of Δ and $L = x_1 + \cdots + x_n$. Then the space of affine 2-stresses can be described as the set $\mathcal{S}_2(\Delta, p)$ of polynomials of degree 2 such that

$$\begin{cases} \prod_{i\in\sigma} x_i \bullet f &= 0 \quad \text{for all} \quad \sigma \notin \Delta \\ \theta_1 \bullet f &= 0 \\ &\vdots \\ \theta_d \bullet f &= 0 \\ L \bullet f &= 0. \end{cases}$$

Now if $f = \sum_i \lambda_{ii} x_i^2 + \sum_{ij\in\Delta} \lambda_{ij} x_i x_j$ is an arbitrary polynomial of degree 2 in $\mathcal{S}^a(\Delta, p)$ we have

1. $L \bullet f = \sum_j 2\lambda_{jj}x_j + \sum_j x_j \sum_{k:\, jk\in\Delta} \lambda_{jk}$, hence
$$\lambda_{jj} = \frac{-1}{2} \sum_{k:\, jk\in\Delta} \lambda_{jk}. \tag{3.8}$$

2. $\theta_i \bullet f = \sum_j 2\lambda_{jj}(p_j)_i x_j + \sum_j x_j \sum_{k:\, jk\in\Delta} \lambda_{jk}(p_k)_i$ for all i. Applying (3.8) we have the following equality for the coefficient of x_j in $\theta_i \bullet f = 0$:
$$-(p_j)_i \sum_{k:\, jk\in\Delta} \lambda_{jk} + \sum_{k:\, jk\in\Delta} \lambda_{jk}(p_k)_i = \sum_{k:\, jk\in\Delta} \lambda_{jk}((p_k)_i - (p_j)_i) = 0. \tag{3.9}$$

Note that (3.3) and (3.9) are exactly the same equations. Since by (3.8) a polynomial in $\mathcal{S}_2^a(\Delta, p)$ is uniquely determined by the coefficients λ_{ij} of squarefree monomials[6], we conclude there is an isomorphism between the space of affine 2-stresses introduced by Lee, and the space of stresses from rigidity theory introduced earlier in this section. In particular, this equivalence shows that Stanley's approach for proving the g-theorem is "dual" (by Macaulay duality) to Kalai's approach in [103].

In this section we focused mostly on the motivations from rigidity theory that are related to topics we have already introduced and discussed in Chapter 2. The reader should not conclude that the only motivation for rigidity theory comes from f-vector theory, as rigidity theory is a very broad and interdisciplinary field.

Remark 3.2.3 (**Differentiation and contraction).** Even though in this section we used mostly differentiation, since that is what Lee used, as was mentioned in Remark 3.1.3, all the equalities would still hold even if we used contraction. In future sections and chapters we will go back to using contraction.

Remark 3.2.4 (**Beyond l.s.o.p.s).** It should be noted that none of the commutative algebra in this section depends on the fact that $\theta_1, \ldots, \theta_d$ are linear forms. Since Macaualy duality would apply for *any* s.o.p., it is very natural to ask what are "stresses" with respect to nonlinear s.o.p.s. This is a hard question, as it is not even known how to generate (non-general) s.o.p.s for arbitrary Stanley-Reisner ideals (even Cohen-Macaulay, or Gorenstein ones).

[6]In [116], Lee shows that this is true in any degree.

3.3 A geometric insight: unexpected hypersurfaces

We now introduce the last perspective we need in order to motivate the notion of an unexpected s.o.p, which will be introduced in Section 3.4. As was mentioned in Chapter 1, even though we have only mentioned (combinatorial) applications of an algebra *satisfying* the WLP/SLP, in other areas such as classical algebraic geometry and differential geometry, it is often the case that it is the *failure* of WLP/SLP that implies interesting behaviour. In this section we will recall some of the notions from the theory of unexpected hypersurfaces and its relations to the failure of SLP of certain algebras. We will come back to this geometric idea of "failing for a reason" in Chapter 5 with a different motivation for studying algebras failing the SLP.

Given a set of points $X_1, \ldots, X_s$ in projective space and integers $\alpha_1, \ldots, \alpha_s > 0$, a classical problem in algebraic geometry is to study hypersurfaces of a given degree passing through X_i with multiplicity at least α_i. Algebraically, this problem can be seen as studying the graded pieces of the ideal $I = I_{X_1}^{\alpha_1} \cap \cdots \cap I_{X_s}^{\alpha_s}$, where I_{X_i} is the defining ideal of the point X_i. Problems similar to the one just described are called *interpolation problems*.

One of the many questions that one can ask regarding the interpolation problem above (and its variations) is the dimension of the space of forms of degree d that vanish at points $X_1, \ldots, X_s$ and at an extra general point P with multiplicity m. In this case, Cook, Harbourne, Migliore and Nagel [32] introduced the notions of "expected" and "actual" dimensions for this space of forms, and introduced the concept of "unexpectedness" exactly when the expected and actual dimension do not match. More concretely, a set of points $X = \{X_1, \ldots, X_s\} \subset \mathbb{P}^n$ admits an *unexpected hypersurface* (or *unexpected curve*, if $n = 2$) of degree d with a general point P of multiplicity m if the actual dimension $\dim(I_X \cap I_P^m)_d$ is strictly larger than the expected dimension $\max(0, \dim(I_X)_d - \binom{m+n-1}{n})$.

Since the definition of unexpected hypersurfaces in 2018, several connections to other areas of mathematics have been found. The connection most relevant to us was in fact already mentioned in [32, Theorem 7.5]. In 1995 Emsalem and Iarrobino [48] showed that in order to study the dimension of graded pieces of ideals of the form $I_{X_1}^{\alpha_1} \cap \cdots \cap I_{X_s}^{\alpha_s}$, one may use the theory of inverse systems to study graded pieces of an algebra defined by an ideal generated by powers of linear forms that are dual to the points $X_1, \ldots, X_s$ instead. See [32, Theorem 7.2] for the exact statement of Theorem 3.3.1.

Theorem 3.3.1 ([48]). *Let $I_1, \dots, I_m$ be ideals of distinct points in $\mathbb{P}^{n-1}$. Choose positive integers $a_1, \dots, a_m$ and let $(l_1^{a_1}, \dots, l_m^{a_m}) \subset R = \mathbb{K}[x_1, \dots, x_n]$ be the ideal generated by powers of the linear forms that are dual to the points of $I_1, \dots, I_m$. Then for any integer $j \geq \max(a_i)$,*

$$\dim\left(\frac{R}{(l_1^{a_1}, \dots, l_m^{a_m})}\right)_j = \dim(I_1^{j-a_1+1} \cap \dots \cap I_m^{j-a_m+1})_j.$$

Note that Theorem 3.3.1 implies we should expect a connection between Lefschetz properties of algebras defined by powers of linear forms and unexpected hypersurfaces. This is because studying the WLP/SLP of $A = R/(l_1^{a_1}, \dots, l_m^{a_m})$ is equivalent to studying the Hilbert function of A/L^k, where L is a general linear form. Indeed, the authors in [32] use Theorem 3.3.1 to show the following:

Theorem 3.3.2 ([32, Theorem 7.5]). *Let $f = L_1 \dots L_d$ be the defining polynomial of a union of lines in $\mathbb{P}^2$ and let Z be the set of points in $\mathbb{P}^2$ dual to these lines. Then Z has an unexpected curve of degree $j+1$ if and only if $A = R/(L_1^{j+1}, \dots, L_d^{j+1})$ fails the SLP due to the map $\times L^2 : A_{j-1} \to A_{j+1}$ failing to have full rank for every general linear form L.*

Unexpected hypersurfaces will not play a role in this text. Instead, the important aspect of the theory of unexpected hypersurfaces that is relevant to us, is the big picture idea that failure of WLP/SLP of a family of algebras of a certain form can be explained geometrically after applying the theory of inverse systems. More concretely, unexpected hypersurfaces can be seen as an example of the following framework:

1. Find a family of algebras with interesting behaviour with respect to Lefschetz properties (algebras defined by powers of linear forms in the case of unexpected hypersurfaces),

2. apply the theory of inverse systems (Theorem 3.3.1 for unexpected hypersurfaces)

3. find an interesting geometric phenomenon that is explained by failure of WLP/SLP of the family in 1.

Our goal for the rest of this chapter is to modify the items above to a combinatorial setting. Note that in practice we are going to start from 3. then go to 2. and then finally 1.

3.4 From points to complexes: a concept of unexpectedness for simplicial complexes

We now focus on introducing unexpected systems of parameters of (Stanley-Reisner ideals of) simplicial complexes. This is the main object we define in this chapter.

Up until now, we have motivated several questions in combinatorial commutative algebra from questions in f-vector theory. An important step that is always present is to use a (often linear) s.o.p. of the Stanley-Reisner ideal of a simplicial complex Δ. In Section 3.2 we recalled a method of generating s.o.p.s for a polytopal simplicial sphere Δ, by considering a convex realization of Δ. From now on we say a sequence of elements is a s.o.p. of Δ if it is a s.o.p. of I_Δ. The broad question we are interested in studying in this chapter is the following.

Question 3.4.1. *Let Δ be a Cohen-Macaulay complex. How can we generate combinatorially interesting s.o.p.s of Δ?*

From a more algebraic perspective, the question we are interested in is the following:

Question 3.4.2. *Let Δ be a d-dimensional Cohen-Macaulay complex, $f \in R$ a homogeneous polynomial and $t, a > 1$ positive integers. When can we guarantee the existence of a s.o.p. $\theta_1, \dots, \theta_{d+1}$ of Δ such that*

1. $x_1^a, \dots, x_n^a, f \in I_\Delta + (\theta_1, \dots, \theta_{d+1})$,
2. $\deg \theta_1 + \cdots + \deg \theta_{d+1} + \deg h_\Delta(x) - d - 1 = t$

In this text we will always take f in Question 3.4.2 to be $L = x_1 + \cdots + x_n$. Under this assumption, it is clear for some pairs (a, t) that such a s.o.p. exists, namely when $a \gg t$, since we can take $\theta_1 = L$ and in general t is the socle degree of $R/(I_\Delta, \theta_1, \dots, \theta_{d+1})$. The interesting scenario is the opposite, when $t \gg a$.

The connection between Question 3.4.2 and the theory of inverse systems comes from the following statements from [93], but we believe they were known before. We note that the matrices below are chosen with respect to a suitable monomial bases.

Lemma 3.4.3 ([93, Lemma 5.3])**.** *Let $f \in R = \mathbb{K}[x_1, \dots, x_n]$, $S = \mathbb{K}[y_1, \dots, y_n]$ and $J \subset R$ a monomial ideal. Then the matrix M that represents the map*

$$\times f^\top : R_i \to R_{i-\deg f}$$

is the same matrix that represents the map

$$f\circ : S_{-i} \to S_{\deg f - i}.$$

In particular, the matrix that represents the map

$$\times f^{\top} : (R/J)_i \to (R/J)_{i-\deg f}$$

is obtained from M by deleting rows and columns corresponding to monomials in J.

Proof. We know multiplication by x_k corresponds to increasing the power of x_k in every m_j by 1. The transpose of this map then satisfies:

$$\times x_k^{\top}(x_1^{a_1}\dots x_n^{a_n}) = \begin{cases} x_1^{a_1}\dots x_k^{a_k-1}\dots x_n^{a_n} & \text{if } a_k > 0 \\ 0 \quad \text{otherwise} \end{cases}.$$

In particular, since $\times f^T$ is a composition and sum of the maps above, given the isomorphism $\varphi : R \to S$ where $\varphi(x_j) = y_j$, we have $\varphi \circ (\times f^{\top}) \circ \varphi^{-1}$ is the same map as $f\circ : S_{-i} \to S_{-i+\deg f}$. Noticing that the matrix that represents the map φ is just the identity matrix, the result follows.

The last part of the statement follows since taking a quotient of R by a monomial ideal J does not add any relation besides removing elements from the basis of each graded piece of R. $\square$

Theorem 3.4.4 (Lefschetz properties of monomial ideals via inverse systems, [93, Theorem 5.3]**).** *Let*

$$J = I + (x_1^{a_1}, \dots, x_n^{a_n}) \subset R = \mathbb{K}[x_1, \dots, x_n]$$

where I is a monomial ideal. Let $L = x_1 + \cdots + x_n$ be a linear form and $A = \frac{R}{J}$. Let $S = \mathbb{K}[y_1, \dots, y_n]$ be a polynomial ring where R acts on S by contraction. Then a multiplication map

$$\times L : A_{i-1} \to A_i$$

fails to be surjective if and only if there exists a sequence of elements $f_1, \dots, f_d \in R$ such that $I + (f_1, \dots, f_d, L)$ is an artinian ideal, and a polynomial $G \in (I + (L, f_1, \dots, f_d))^{-1}_{-i}$

such that

$$G = \sum_j c_j y_1^{b_{j1}} \dots y_n^{b_{jn}},$$

$c_j \in \mathbb{K}$ *and* $b_{jk} < a_k$ *for every* j, k.

Proof. The multiplication map $\times L : A_{i-1} \to A_i$ fails to be surjective if and only if its transpose fails to be injective. By Lemma 3.4.3, a nonzero polynomial G in the kernel of $\times L^\top : A_i \to A_{i-1}$ exists if and only if there exists a polynomial G' satisfying $m \circ G' = L \circ G' = x_i^{a_i} \circ G' = 0$ for every i, and every generator m of I. Taking any such G' and the Gorenstein artinian ideal P such that $P^{-1} = (G')$, we conclude $I + L \subset P = (f_1, \dots, f_d)$, hence the generating set $(f_1, \dots, f_d)$ of P can be taken as the sequence of elements from the statement. The assumption on the powers of variables in the monomials appearing in G guarantees $x_i^{a_1} \in P$ for every i. $\square$

Theorem 3.4.4 is the connection between Lefschetz properties and inverse systems that we will constantly use in our setting, instead of the results in [48], which are extremely useful in the context of algebras defined by powers of linear forms.

We are now ready to define unexpected systems of parameters.

Definition 3.4.5 (Unexpected systems of parameters). Let Δ be a d-dimensional Cohen-Macaulay simplicial complex over some field $\mathbb{K}$ on vertex set $[n]$, $f \in R = \mathbb{K}[x_1, \dots, x_n]$ a homogeneous polynomial, $t \geq \deg h_\Delta(x)$ an integer and $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$. We say a sequence of elements $\theta_1, \dots, \theta_{d+1} \in R$ is an $(f, \mathbf{a})$*–unexpected system of parameters of total degree* t if:

(U1) $\theta_1, \dots, \theta_{d+1}$ is a homogeneous system of parameters of I_Δ,

(U2) $\deg \theta_1 + \dots + \deg \theta_{d+1} + \deg h_\Delta(x) - d - 1 = t$,

(U3) $f \in I_\Delta + (\theta_1, \dots, \theta_{d+1})$,

(U4) $x_i^{a_i} \in I_\Delta + (\theta_1, \dots, \theta_{d+1})$ for every i, and

(U5) $\mathrm{HF}_{R/J}(t) \leq \mathrm{HF}_{R/J}(t - \deg f)$, where $J = I_\Delta + (x_1^{a_1}, \dots, x_n^{a_n})$.

When f is the sum of variables of R and $\mathbf{a} = (a, \dots, a)$, we simply say $\theta_1, \dots, \theta_{d+1}$ is an a*-unexpected system of parameters of total degree* t.

One possible interpretation for unexpected s.o.p.s, is that if an algebra A defined by an ideal $J = I_\Delta + (x_1^a, \dots, x_n^a)$, where I_Δ is Gorenstein, fails the WLP due to surjectivity in some degree $t-1$, then a polynomial causing the failure of WLP is the Macaualy dual generator of an artinian Gorenstein algebra which (as we will see) in many cases is defined by an ideal of the form $I_\Delta + (\theta_1, \dots, \theta_{d+1})$. A s.o.p. $\theta_1, \dots, \theta_{d+1}$ of Δ arising in this way is an unexpected s.o.p.. The proposition below makes the connection between failure of WLP and existence of unexpected s.o.p.s explicit.

Proposition 3.4.6. *Let Δ be a d-dimensional Cohen-Macaulay simplicial complex over a field $\mathbb{K}$ on vertex set $[n]$ and let $L = x_1 + \cdots + x_n$. Assume $\theta_1, \dots, \theta_{d+1}$ is an $(L, \mathbf{a})$–unexpected s.o.p. of Δ of total degree t, where $\mathbf{a} = (a_1, \dots, a_n)$. Then the monomial algebra*

$$A = \frac{R}{I_\Delta + (x_1^{a_1}, \dots, x_n^{a_n})}$$

fails the WLP due to surjectivity in degree $t-1$.

Proof. By (U2) the socle degree of the algebra $\frac{R}{I_\Delta + (\theta_1, \dots, \theta_{d+1})}$ is t. Let F be an element of the inverse system of $I_\Delta + (\theta_1, \dots, \theta_{d+1})$ of degree t. By (U3) and (U4) we know $L \circ F = x_i^{a_i} \circ F = 0$ and so by Theorem 3.4.4 we conclude the map

$$\times L^\top : A_t \to A_{t-1}$$

is not injective, where $A = \frac{R}{I_\Delta + (x_1^{a_1}, \dots, x_n^{a_n})}$. The result then follows by (U5). $\square$

Many of our results in the rest of this chapter will focus on studying the converse of Proposition 3.4.6.

Remark 3.4.7 (Intersecting kernels). An interesting application from the ideas in this section is that we are able to study the intersection of kernels of maps of the form $\times\theta^\top : A_i \to A_{i-1}$, where θ is a linear form and A is a (monomial) artinian algebra. Examples of this idea can be found in [93, Corollary 5.5, Corollary 5.7], where the class of spheres appearing in the second part of the generalized lower bound conjecture (*stacked spheres*) is used to show that not only do certain maps as above fail to be injective, but the dimension of the intersection of such kernels is nonzero when the linear forms are general.

3.5 A combinatorial meaning to linear unexpectedness: balanced complexes

One missing aspect of the class of unexpected s.o.p.s so far, is a combinatorial interpretation for the concept of unexpectedness. Our goal in this section is to find such an interpretation when the unexpected s.o.p. is linear. More concretely, in this section we will show that the existence of an unexpected linear s.o.p. for a homology sphere is related to the balanced property of a simplicial complex. We first need to recall the definition of balanced complexes.

Definition 3.5.1 (**Balanced complexes**)**.** Let Δ be a d-dimensional simplicial complex on vertex set V. A *coloring* ρ of Δ on n colors is a map $\rho : V \to [n]$ such that if $\{u, v\} \in \Delta$, then $\rho(u) \neq \rho(v)$. We say Δ is *balanced* if there exists a coloring ρ of Δ on $d+1$ colors. We will always assume that a balanced complex Δ comes equipped with a coloring ρ.

The class of balanced complexes[7] was introduced by Stanley [184] in 1979. As is pointed out in [184], Stanley's original goal with defining balanced complexes was to obtain information on their f-vector[8]. In particular, Stanley noticed that for balanced complexes it is possible to define *multigraded face numbers*, which essentially count the number of faces that use a given set of colors. Algebraically, this can be seen from the fact that a coloring ρ of a d-dimensional balanced complex Δ induces a natural grading on R/I_Δ where $\deg x_i = \mathbf{e}_{\rho(i)} \in \mathbb{N}^{d+1}$ and $\mathbf{e}_k$ denotes the k-th element of the canonical basis of $\mathbb{N}^{d+1}$ (see [184, Section 3] for more details).

One of Stanley's first results on balanced complexes was related to a special s.o.p. of balanced complexes that is now called the *colored system of parameters*.

Proposition 3.5.2 ([184, Corollary 4.2])**.** *Let Δ be a d-dimensional balanced complex with coloring ρ and let*

$$\theta_i = \sum_{\rho(j)=i} x_j \quad \textit{for all} \quad i = 1, \dots, d+1.$$

Then $\theta_1, \dots, \theta_{d+1}$ forms a s.o.p. of Δ. This s.o.p. is called the colored s.o.p. *of Δ with respect to ρ.*

Note that in the finer grading of R/I_Δ where $\deg x_i = \mathbf{e}_{\rho(i)}$, the linear forms θ_i above are still homogeneous with degree $\deg \theta_i = \mathbf{e}_i$. In particular the ring $R/(I_\Delta +$

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[7] In the literature, a balanced complex (as we defined it) is sometimes called "completely balanced".

[8] It should be noted that [184] was published only a couple of years after Stanley's proof of the UBC that characterized the possible f-vectors of Cohen-Macaulay complexes.

$(\theta_1, \ldots, \theta_{d+1}))$ is also $\mathbb{N}^{d+1}$-graded. In the same paper, Stanley also showed that $x_i^2 \in I_\Delta + (\theta_1, \ldots, \theta_{d+1})$ for every $i = 1, \ldots, d+1$. We will soon give a new proof of this statement using Macaulay duality. Before proving it, we first need some preliminary results and definitions.

Definition 3.5.3 (Facet-ridge graphs). Let Δ be a pseudomanifold without boundary[9]. The *facet-ridge graph* of Δ is the graph $G(\Delta)$, where the vertices of $G(\Delta)$ are the facets of Δ, and two vertices σ_1, σ_2 of $G(\Delta)$ are adjacent if and only if $\sigma_1 \cap \sigma_2$ is a ridge of Δ.

Note that since pseudomanifolds are strongly connected, the facet-ridge graphs we consider in this text are always connected.

The result we need is about the structure of facet-ridge graphs of balanced $\mathbb{K}$-homology spheres. The statement we include here is a combination of results due to Klee and Novak [110], and a result of Joswig [100] (see also [107, p. 22]). We include a proof of this statement for the sake of completeness, and refer the reader to [110] for the definition of normal pseudomanifolds that will only appear in the proof below.

Theorem 3.5.4 ([110, Lemma 6.2 1.],[100, Proposition 11])**.** *Let Δ be a balanced $\mathbb{K}$-homology sphere. Then the facet-ridge graph of Δ is bipartite. Moreover, if Δ is a simplicial sphere, then Δ is balanced if and only if the facet-ridge graph of Δ is bipartite.*

Proof. The first part of the statement follows directly from [110, Lemma 6.2.1] since homology spheres are orientable normal pseudomanifolds.

For the second part of the statement, notice first that if $d = 1$, then both Δ and $G(\Delta)$ are cycle graphs on the same number of vertices, and so the result holds. Assume now that $d > 1$ and notice that if Δ is a simplicial d-sphere, by definition it is homeomorphic to a d-sphere and hence simply connected. Assuming the facet-ridge graph of Δ is bipartite, we conclude the facet-ridge graph of $\mathrm{link}_\Delta(\sigma)$ is bipartite for every $\sigma \in \Delta$. The result then follows directly by [100, Proposition 11]. □

Next, in order to relate the balanced property to the existence of unexpected s.o.p.s, we need the following statement due to Dao and Nair [40]. Recall that the *(signless) incidence matrix* of a graph G with vertex set V and edge set E is the $|V| \times |E|$ matrix $M(G)$ such

[9]It is possible to define the facet-ridge graph of any pseudomanifold, but we will not need that level of generality in this text.

that

$$M(G)_{ij} = \begin{cases} 1 & \text{if} \quad i \in j \\ 0 & \text{otherwise..} \end{cases}$$

Theorem 3.5.5 ([40, Section 4])**.** *Let $d > 0$, Δ a d-dimensional pseudomanifold without boundary on vertex set $[n]$, $L = x_1 + \cdots + x_n \in R = \mathbb{K}[x_1, \ldots, x_n]$ and $A = R/J$ where $J = I_\Delta + (x_1^2, \ldots, x_n^2)$. Then*

(1) The matrix that represents the multiplication map $\times L : A_d \to A_{d+1}$ is the transpose of $M(G(\Delta))$.

(2) The map $\times L : A_d \to A_{d+1}$ has full rank if and only if $G(\Delta)$ is not bipartite.

In order to explain how we are going to apply Theorem 3.5.5, we first need the following two well-known results.

Lemma 3.5.6. *(1) Let G be a bipartite connected graph on vertex set V and bipartition $V = B_1 \cup B_2$. Then $\operatorname{rank} M(G) = |V| - 1$, and if G is not a tree, the kernel of $M(G)^\top$ is spanned by the vector*

$$\sum_{i \in B_1} \mathbf{e}_i - \sum_{i \in B_2} \mathbf{e}_i.$$

(2) Let Δ be a d-dimensional pseudomanifold without boundary, and f_d facets and f_{d-1} ridges. Then

$$f_d = \frac{2}{d+1} f_{d-1}.$$

In particular, if $d > 0$ then $f_d \leq f_{d-1}$.

Proof. (1) See [66, Theorem 8.2.1] for a proof.

(2) Consider the set of tuples $X = \{(\tau, \sigma) \colon |\tau| + 1 = |\sigma| = d + 1 \quad \text{and} \quad \tau \subset \sigma\}$. Since every facet σ contains exactly $d + 1$ distinct ridges, we conclude $|X| = (d + 1) f_{d+1}$. On the other hand, since Δ is a pseudomanifold without boundary we know every ridge τ is contained in exactly two distinct facets, hence $|X| = 2 f_{d-1}$. Combining both equalities we conclude $(d + 1) f_d = 2 f_{d-1}$.

□

We now have all the required pieces to prove the main results of this section.

First, it is clear that for any complex Δ, the only monomials not in $I_\Delta + (x_1^2, \dots, x_n^2)$ are exactly the monomials corresponding to faces of Δ, hence following notation from Theorem 3.5.5 we have $\dim A_i = f_{i-1}$, the number of $(i-1)$–faces of Δ. For $d > 0$, by Lemma 3.5.6 (2) we conclude $\dim A_d \geq \dim A_{d+1}$. This inequality should be seen as the fifth condition (U5) in the definition of unexpected s.o.p.s for $a = 2$.

Next, note that condition (U2) describes the socle degree of the algebra defined by modding an unexpected s.o.p. in the face ring of Δ. In the case where the unexpected s.o.p. is linear and Δ is a d-dimensional homology sphere, we have $\deg \theta_i = 1$ and $\deg h_\Delta(x) = d+1$. These two facts combined give us that t should be equal to $d+1$.

These two remarks together with Theorem 3.5.5 (2) and Proposition 3.4.6 show that in order for a d-dimensional homology sphere Δ to have an 2-unexpected s.o.p., it must be balanced.

Lemma 3.5.7 (**The Macaulay dual generator of a colored s.o.p.,** [94, Lemma 4.3])**.** *Let $\Delta = \langle \sigma_1, \dots, \sigma_s \rangle$ be a balanced d-dimensional $\mathbb{K}$-homology sphere on vertex set $[n]$, ρ a coloring of Δ and $\theta_1, \dots, \theta_{d+1}$ the corresponding colored s.o.p.. Then the Macaulay dual generator of $I_\Delta + (\theta_1, \dots, \theta_{d+1})$ is the polynomial*

$$F_{\Delta,\rho} = \sum_{\sigma \in B_1} x_\sigma - \sum_{\sigma \in B_2} x_\sigma,$$

where $x_\sigma = \prod_{i \in \sigma} x_i$ and $B_1 \cup B_2$ is a bipartition of $G(\Delta)$.

Proof. First note that since Δ is a balanced homology sphere, Theorem 3.5.4 implies $G(\Delta)$ is bipartite and hence it is posible to partition $V(G(\Delta))$ into two disjoint independent sets B_1, B_2. Since we know $I_\Delta + (\theta_1, \dots, \theta_{d+1})$ defines an artinian Gorenstein algebra of socle degree $d+1$, we only need to find a homogeneous polynomial F of degree $d+1$ such that $m \circ F = \theta_i \circ F = 0$ for every i and every generator m of I_Δ. The result would then follow since Macaulay duality gives a bijection between artinian Gorenstein algebras of the form R/I and homogeneous polynomials in $R = \mathbb{K}[x_1, \dots, x_n]$.

By the definition of $F_{\Delta,\rho}$ we have $m \circ F_{\Delta,\rho} = 0$ for every generator m of I_Δ. Hence we only need to show that $\theta_i \circ F_{\Delta,\rho} = 0$ for every i. Since the support of every monomial in $F_{\Delta,\rho}$ is a face of Δ, we may view $F_{\Delta,\rho}$ as an element of $A = \frac{R}{I_\Delta + (x_1^2, \dots, x_n^2)}$. Finally, by Lemma 3.4.3 we only need to show $F_{\Delta,\rho}$ is in the kernel of the map $\times\theta_i^\top : A_{d+1} \to A_d$ for every i.

By the definition of the colored s.o.p. of Δ we have $L = x_1+\cdots+x_n = \theta_1+\cdots+\theta_{d+1}$. By Theorem 3.5.5 (1) the matrix M representing the map $\times L^\top : A_{d+1} \to A_d$ is equal to the signless incidence matrix of $G(\Delta)$. As a consequence, since $G(\Delta)$ is a connected bipartite graph, Lemma 3.5.6 (1) implies $F_{\Delta,\rho} \in \ker M$. Since every facet of Δ contains exactly one vertex of each color, we conclude for every ridge τ there exists a unique color c_τ such that $\tau \cap \rho^{-1}(c_\tau) = \emptyset$.

Next, if $\times\theta_i^\top(F_{\Delta,\rho}) = 0$ for all i we are done. Assume without loss of generality that $\times\theta_1^\top(F_{\Delta,\rho}) \neq 0$, and let τ be a ridge of Δ such that $x_\tau = \prod_{j\in\tau} x_j$ is a monomial of $\times\theta_1^\top(F_{\Delta,\rho})$ with a nonzero coefficient. Since τ contains vertices of every color other than 1, we conclude the coefficient of x_τ in $\times\theta_i^\top(F_{\Delta,\rho})$ is zero for every $i \neq 1$. This is a contradiction since

$$\times\theta_1^\top(F_{\Delta,\rho}) + \cdots + \times\theta_{d+1}^\top(F_{\Delta,\rho}) = \times L^\top(F_{\Delta,\rho}) = 0$$

and $\times\theta_1^\top(F_{\Delta,\rho})$ cannot cancel with any other term in the sum above. □

We are now ready to prove the main result of this section.

Theorem 3.5.8 (2**-Unexpected systems of parameters and balanced simplicial spheres).** *Let Δ be a d-dimensional simplicial sphere with $d > 0$. Then Δ has an 2-unexpected system of parameters if and only if Δ is balanced. In this case, the colored s.o.p. is an example of an 2-unexpected s.o.p. of Δ.*

Proof. Let $\theta_1, \ldots, \theta_{d+1}$ be the colored s.o.p. of Δ given by a coloring ρ of Δ and let L be the linear form given by the sum of variables of $R = \mathbb{K}[x_1, \ldots, x_n] \supset I_\Delta$. Assume first that Δ is balanced. Note that $L = \theta_1 + \cdots + \theta_{d+1}$, hence $L \in I_\Delta + (\theta_1, \ldots \theta_{d+1})$. Lemma 3.5.7 directly shows $x_i^2 \in I_\Delta + (\theta_1, \ldots, \theta_{d+1})$ for every i since $x_i^2 \circ F_{\Delta,\rho} = 0$. The only missing property to conclude $\theta_1, \ldots, \theta_{d+1}$ is an 2-unexpected s.o.p. is that $\mathrm{HF}_{R/J}(d+1) \leq \mathrm{HF}_{R/J}(d)$, where $J = I_\Delta + (x_1^2, \ldots, x_n^2)$. This follows by Lemma 3.5.6 (2), and since $\mathrm{HF}_{R/J}(d+1) = f_d$ and $\mathrm{HF}_{R/J}(d) = f_{d-1}$.

Assume now that Δ has an 2-unexpected system of parameters $\theta_1, \ldots, \theta_{d+1}$ of total degree $t \geq d + 1$. Then by definition we know:

(U1) $I_\Delta + (\theta_1, \ldots, \theta_{d+1})$ is an artinian Gorenstein ideal and hence has a Macaulay dual generator F,

(U3) $L \in I_\Delta + (\theta_1, \ldots, \theta_{d+1})$,

(U4) $x_i^2 \in I_\Delta + (\theta_1, \ldots, \theta_{d+1})$, and

(U5) $\mathrm{HF}(t, R/J) \leq \mathrm{HF}(t-1, R/J)$, where $J = I_\Delta + (x_1^2, \ldots, x_n^2)$.

By (U5) above and since the socle degree of R/J is $d+1$ we conclude $t \leq d+1$, but since by definition we have $t \geq d+1$, we must have $t = d+1$. Since $\theta_1, \ldots, \theta_{d+1}$ is a s.o.p. of I_Δ, it is a well-known fact (see for example [93, Lemma 3.5]) the Hilbert series of $\frac{R}{I_\Delta + (\theta_1, \ldots, \theta_{d+1})}$ is equal to

$$h_\Delta(x) \prod_{i=1}^{d+1} (1 + x + \cdots + x^{\deg \theta_i - 1}).$$

In particular, $\deg h_\Delta(x) + \deg \theta_1 + \cdots + \deg \theta_{d+1} - d - 1 = d+1$, and since $\deg h_\Delta(x) = d+1$ we conclude $\theta_1, \ldots, \theta_{d+1}$ is a linear system of parameters of I_Δ. By Theorem 3.4.4 we know the existence of the polynomial F says the algebra R/J fails the WLP in degree $\deg F = d+1$ due to surjectivity. By Theorem 3.5.5 (2) we conclude the facet-ridge graph of Δ is bipartite and finally by Theorem 3.5.4 we conclude Δ is balanced. $\square$

Theorem 3.5.8 shows that 2-unexpectedness can be seen as an algebraic way of defining the balanced property for simplicial spheres. The reason we are not able to make the weaker (and more natural) assumption in Theorem 3.5.8 that Δ is a homology sphere is because on the combinatorial side, more specifically in the work of Joswig [100], the fact that a simplicial sphere is simply-connected plays a role in the proofs, and as was mentioned in Section 2.2, not every homology sphere is simply connected. We strongly believe, however, that it is possible to replace the simplicial sphere assumption from Theorem 3.5.8 by homology sphere.

Finally, the relationship between 2-unexpected s.o.p.s and the balanced property is even stronger than the conclusion of Theorem 3.5.8. The intuition for the stronger connection between the two topics can be seen from Lemma 3.5.6 (1): the kernel of the last multiplication map is one dimensional (over a field of characteristic 0). In particular, since any Macaulay dual generator of an 2-unexpected s.o.p. of a homology sphere would have to be in the same kernel, the restriction on the dimension of the kernel allows us to show the following result:

Corollary 3.5.9 (A characterization of 2-unexpected systems of parameters). *Let Δ be a d-dimensional balanced simplicial sphere and $\theta'_1, \dots, \theta'_{d+1}$, $\theta''_1, \dots, \theta''_{d+1}$ be two 2-unexpected s.o.p.s of Δ over a field of characteristic 0. Then*

$$I_\Delta + (\theta'_1, \dots, \theta'_{d+1}) = I_\Delta + (\theta''_1, \dots, \theta''_{d+1}).$$

In particular, we can take $\theta'_1, \dots, \theta'_{d+1}$ to be the colored s.o.p. of Δ.

Proof. Let $A = R/J$, where $J = I_\Delta + (x_1^2, \dots, x_n^2)$ and $L = x_1 + \cdots + x_n$. By Theorem 3.5.5 (1) and Lemma 3.5.6 (1) we know $\dim \ker \times L^\top : A_{d+1} \to A_d = 1$. Let ρ be a coloring of the 1-skeleton of Δ, $\theta_1, \dots, \theta_{d+1}$ the corresponding colored s.o.p. and $F_{\Delta,\rho}$ the Macaulay dual generator of $I_\Delta + (\theta_1, \dots, \theta_{d+1})$. By Theorem 3.5.8 we know this colored s.o.p. is an 2-unexpected s.o.p. of Δ, and by Theorem 3.4.4 we know F_ρ generates the kernel of $\times L^\top : A_{d+1} \to A_d$. Moreover, for every other 2-unexpected s.o.p. of Δ, the Macaulay dual generator G of the corresponding ideal I is also in the same kernel, and as the dimension of the kernel is 1 we conclude $G = cF_\rho$ for some constant c. Macaulay duality then implies $I_\Delta + (\theta_1, \dots, \theta_{d+1}) = I$. □

For the last result of this section, we will need a result of Hausel [81], which we will only cite for now, but we will state it properly in later chapters (see Theorem 5.1.3).

Corollary 3.5.10 (Linear unexpectedness is the same as 2-unexpectedness). *Let Δ be a d-dimensional $\mathbb{K}$-homology sphere where $d > 0$ and $\mathbb{K}$ is a field of characteristic 0. Then Δ admits an a-unexpected linear s.o.p. if and only if it admits an 2-unexpected s.o.p..*

Proof. If Δ admits an 2-unexpected s.o.p. $\theta_1, \dots, \theta_{d+1}$, then we know $I_\Delta + (x_1^2, \dots, x_n^2) \subset I_\Delta + (\theta_1, \dots, \theta_{d+1})$. In particular since the socle degree of $\frac{R}{I_\Delta + (x_1^2, \dots, x_n^2)}$ is $d+1$, we conclude the socle degree of $A = \frac{R}{I_\Delta + (\theta_1, \dots, \theta_{d+1})}$ is at most $d+1$. By (U2) the total degree t of the unexpected s.o.p. $\theta_1, \dots, \theta_{d+1}$ is equal to the socle degree of A, and by definition we know $t \geq d+1$. We then conclude $d+1 \leq t \leq d+1$, and hence $t = d+1$. As a consequence, we have

$$\deg \theta_1 + \cdots + \deg \theta_{d+1} = d+1,$$

and so each θ_i must be a linear form, so the unexpected s.o.p. must be linear.

Conversely, if Δ admits an a-unexpected linear s.o.p. $\theta_1, \dots, \theta_{d+1}$, then by the same argument as above we know the total degree of the unexpected s.o.p. is $d+1$. Let $A_a = \frac{R}{I_\Delta + (x_1^a, \dots, x_n^a)}$. By (U5) we conclude

$$\mathrm{HF}_{A_a}(d+1) \leq \mathrm{HF}_{A_a}(d). \tag{3.10}$$

Since Δ is pure of dimension d, the algebra A_a is level of socle degree $(a-1)(d+1)$. Hausel's result [81, Theorem 6.2] then says the multiplication maps $\times L : (A_a)_i \to (A_a)_{i+1}$, where $L = x_1 + \cdots + x_n$, are injective for all $i = 0, \dots, \lfloor ((a-1)(d+1)-1)/2 \rfloor$. Noting that for $a > 2$ we have

$$d < d + \frac{1}{2} = \frac{(3-1)(d+1)-1}{2} \leq \frac{(a-1)(d+1)-1}{2},$$

we conclude for $a > 2$ the multiplication map $\times L : (A_a)_d \to (A_a)_{d+1}$ is always injective, and hence

$$\mathrm{HF}_{A_a}(d+1) \geq \mathrm{HF}_{A_a}(d). \tag{3.11}$$

Combining (3.10) and (3.11) we conclude $\mathrm{HF}_{A_a}(d+1) = \mathrm{HF}_{A_a}(d)$, so the injectivity of the map $\times L : (A_a)_d \to (A_a)_{d+1}$ implies the map is also surjective. This is a contradiction by Proposition 3.4.6, so we concude $a = 2$. $\square$

3.6 Elementary symmetric polynomials: *a* "universal" unexpected system of parameters

We now introduce a second family of unexpected s.o.p.s arising from combinatorics. The results in this section are taken from [93]. Here, we only motivate some of the main statements from [93]. The techniques used and the motivation of the other main results will be presented in the next chapter in a slightly different setting.

So far in this text, we have described several problems in algebraic, enumerative, geometric and topological combinatorics, but there is a big part of combinatorics that we have not mentioned yet: symmetry.

One of the most important themes in combinatorics is the use of symmetry. In geometric combinatorics for example, this can be seen in the study of centrally symmetric spheres, which are special simplicial spheres that admit an involution on its vertices[10]. In algebraic combinatorics, symmetry plays a very important role (for example) through the study of *symmetric polynomials*, which are polynomials f on n variables such that $f(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$ for every permutation π of $[n]$.

An important class of symmetric polynomials is the family of *elementary symmetric polynomials*: $e_1, \ldots, e_n \in R = \mathbb{K}[x_1, \ldots, x_n]$, where the polynomial

$$e_i = \sum_{j_1 < \cdots < j_i} x_{j_1} \ldots x_{j_i}$$

is called the i-th elementary symmetric polynomial. The polynomials $e_1, \ldots, e_n$ are well-known to generate the ring of all symmetric polynomials on n variables (see [172, Theorem H.1]), which is an important subring of the polynomial ring R called the *invariant ring* of the symmetric group.

The property of of symmetric polynomials that is relevant to us is in some sense almost opposite to the notion of invariant rings. The *coinvariant ring* of the symmetric group is the polynomial ring defined by the quotient of the ideal $(e_1, \ldots, e_n)$. Among the many interesting properties of this ring, it can be shown that $R/(e_1, \ldots, e_n)$ is the cohomology ring of the *flag variety*, which is an important variety in algebraic geometry, but we will not need its definition here (see [80, Exercise 4, Section 4.D] for the proof over the integers). Since $R/(e_1, \ldots, e_n)$ is an artinian ring defined by n elements on n variables $(e_1, \ldots, e_n)$ is in fact a complete intersection.

In 2021, Herzog and Moradi [85] noticed[11] that not only is $(e_1, \ldots, e_n)$ an R-regular sequence, but the subset $e_1, \ldots, e_{d+1}$ is a system of parameters for *every* R/I_Δ such that Δ is a d-dimensional simplicial complex. Below we give a (very simple) proof of this statement in a slightly more general setting, which we believe emphasizes the applications of this fact.

Lemma 3.6.1 ([93, Lemma 3.6]). *Let $I = (f_1, \ldots, f_n) \subset R = \mathbb{K}[x_1, \ldots, x_n]$ be a complete intersection such that for every d-dimensional complex Δ, $f_i \in I_\Delta$ for every $i > d + 1$.*

[10]The involution imposes enough restrictions that the f-vector theory of centrally symmetric (cs) spheres is an active area of research. An important example class of cs-spheres comes from a subclass of Bier spheres, defined in Definition 2.2.8. See [16, Section 6.2] for more details.

[11]It is unclear when was the first time this (simple, yet very interesting) fact was first observed. This fact was also used (sometimes in a slightly different language) for example in the work of De Concini, Eisenbud and Procesi [42], Garsia and Stanton [65], Smith [178] and Adams and Reiner [2].

Then $f_1, \ldots, f_{d+1}$ is a s.o.p. for every I_Δ where Δ is a d-dimensional complex.

Proof. Since I is a complete intersection on $n = \dim R$ generators, we know R/I is a finite dimensional vector space. By assumption we know $I \subseteq I + (f_1, \ldots, f_{d+1})$, hence $R/(I_\Delta + (f_1, \ldots, f_{d+1}))$ is also a finite dimensional vector space for every d-dimensional complex Δ. $\square$

In view of (their version of) Lemma 3.6.1, Herzog and Moradi called the s.o.p. of elementary symmetric polynomials "the" *universal s.o.p.*. In Chapter 4 we will see that this is only *a* s.o.p. with this universal property.

Example 3.6.2 (The (*first example of a*) "universal" system of parameters). Let $R = \mathbb{K}[x_1, \ldots, x_n]$ and Δ be the boundary of the simplex on $[n]$. Note that the Stanley-Reisner ideal of Δ is $I_\Delta = (x_1 \ldots x_n) = (e_n)$, and $\dim \Delta = d = n-2$. In particular, $(e_1, \ldots, e_n) = I_\Delta + (e_1, \ldots, e_{n-1})$, so we may see $e_1, \ldots, e_{n-1}$ as a s.o.p. of $I_\Delta = (e_n)$.

Given any d-dimensional complex Δ, we know $\{i_1, \ldots, i_{d+2}\} \notin \Delta$, hence $e_j \in I_\Delta$ for every $j \geq d+2$. By Lemma 3.6.1 we conclude $(e_1, \ldots, e_{d+1})$ is a s.o.p. for every d-dimensional complex Δ.

A combination of results from [93] (see also [94, Theorem 3.5]) implies the following.

Theorem 3.6.3 ("The" universal s.o.p. is unexpected, [93, Corollary 4.5 and Theorem 6.6]**).** *Let Δ be a $\mathbb{K}$-homology sphere. Then the elementary symmetric polynomials $e_1, \ldots, e_{d+1}$ form an $(d+2)$–unexpected system of paramters of total degree $\binom{d+2}{2}$ of Δ.*

3.7 Collapsibility and the nonexistence of unexpected s.o.p.s

We now focus on finding settings where the *nonexistence* of unexpected s.o.p.s can be guaranteed. One one hand, this is interesting because it shows (as we will soon see) certain topological properties can be seen as obstructions for the existence of unexpected s.o.p.s. On the other hand, our tools for showing a complex does not admit a s.o.p. involve showing certain multiplication maps have full rank, and hence is a step towards proving certain algebras satisfy the WLP. Moreover, the matrices appearing in this section turn out to be related to Rees algebras of certain squarefree monomial ideals.

3.7.1 Incidence complexes and half-hollow edgewise subdivisions

The first objects we need to introduce in this section are called *incidence complexes*, and were introduced by the author in [92, 95].

Definition 3.7.1 (Incidence complexes [92, 95]**).** Let Δ be a simplicial complex of dimension d and $1 \leq i < d$. The i-th *incidence complex* of Δ, denoted by $\Delta(i)$ has as its vertex set the set of $(i-1)$–faces of Δ, and a subset σ of vertices is a facet of $\Delta(i)$ if and only if the vertices of σ are the $(i-1)$–faces of an i-face of Δ.

We note that if F_i, F_j are two facets of an incidence complex, then $|F_i \cap F_j| \leq 1$ (see [92, Proposition 20.2]).

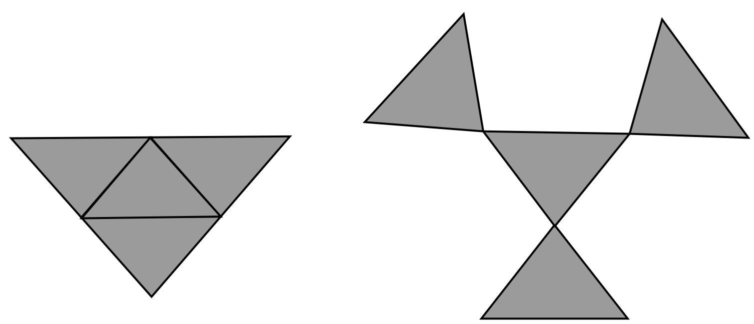


Figure 3.1: A complex Δ on the left, and its incidence complex $\Delta(2)$ on the right.

The original application of incidence complexes in [92, 95] was to show the log-matrices of incidence complexes (with entries over a field $\mathbb{K}$) are equal to multiplication maps of maps of the form $\times L : A_i \to A_{i+1}$, where $L = x_1 + \cdots + x_n$ and

$$A = \frac{\mathbb{K}[x_1, \ldots, x_n]}{I_\Delta + (x_1^2, \ldots, x_n^2)}.$$

Here, our first goal is to show a similar result regarding the algebra defined by $I_\Delta + (x_1^a, \ldots, x_n^a)$ for arbitrary a. When $a = 2$, the matrix M that represents the multiplication map $\times L : A_i \to A_{i+1}$ is $(i+1)$-stochastic: the sum of every row is equal to $i+1$. This follows from the fact that for $m = x_{j_1} \ldots x_{j_{i+1}}$ we have

$$\times L^\top(m) = \sum_{k=1}^{i+1} \frac{m}{x_{j_k}},$$

in other words, the columns of $M^\top$ have $i+1$ nonzero entries (which are all equal to 1). For $a > 2$ the situation is very different. Since nonsquarefree monomials are nonzero in this case, it is easy to find examples where the multiplication map is not represented by a stochastic matrix.

From now on, for a Δ be a d-dimensional complex and $a > 1$ an integer, we set the notation

$$A_\Delta(a) = \frac{R}{I_\Delta + (x_1^a, \ldots, x_n^a)} \quad \text{where} \quad R = \mathbb{K}[x_1, \ldots, x_n].$$

Note that in degree $d(a-1)+1$, every nonzero monomial $m \in A_\Delta(a)$ must satisfy $|\operatorname{supp}(m)| = d+1$. In particular for $L = x_1 + \cdots + x_n$, the sum of entries in every row of the matrix M that represents the multiplication map $\times L : A_\Delta(a)_{d(a-1)} \to A_\Delta(a)_{d(a-1)+1}$ is $d+1$. Our first goal is to show that M is the log-matrix of the facet ideal of a complex associated to Δ, but first we need to introduce some constructions from combinatorial topology and hypergraph theory.

Given a simplicial complex Δ, in 2000 Edelsbrunner and Grayson [45] introduced a complex called the "edgewise subdivision" of Δ. This construction has since then appeared in many distinct scenarios and is often viewed as the algebraic version of barycentric subdivision[12], in part due to its connections to Veronese algebras of face rings [21, 26, 115].

In [149], Mirzakhani and Vondrák introduced a family of hypergraphs based on the edgewise subdivision of a simplex [45], in order to study problems related to hypergraph colorings. Here we define a slightly more general version of their construction, which we call the *r-fold half-hollow edgewise subdivision* of a simplicial complex Δ, denoted by $\operatorname{hesd}(\Delta, r)$.

Let $r > 0$ be an integer, and denote by

$$\Omega_n^r = \{(a_1, \ldots, a_n) \colon a_1 + \cdots + a_n = r \quad \text{and} \quad a_i \geq 0\} \subset \mathbb{R}^n$$

the set of lattice points in the r-th dilation of the standard simplex. Moreover, for a vector $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$, we denote by $\operatorname{supp}(\mathbf{a}) = \{i \colon a_i \neq 0\}$ the support of $\mathbf{a}$. Note that for a simplicial complex Δ on vertex set $[n] = \{1, \ldots, n\}$, we may view the vertices of Δ as the canonical basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of $\mathbb{R}^n$.

[12] The barycentric subdivision of a simplicial complex Δ is obtained by subdividing faces of Δ. More specifically, the new complex Γ has a new vertex for every nonempty face of Δ.

Let $\Delta = \langle F_1, \ldots, F_s \rangle$ be a simplicial complex on vertex set $[n]$ such that $|F_i \cap F_j| \leq 1$. The complex $\operatorname{hesd}(\Delta, r)$ has vertex set Ω_n^r. A collection of vectors $A \subset \Omega_n^r$ is a facet of $\operatorname{hesd}(\Delta, r)$ if and only if there exists a facet F of Δ such that the following conditions are satisfied:

(HESD1) $\bigcup_{\mathbf{b} \in A} \operatorname{supp}(\mathbf{b}) \subseteq F$

(HESD2) $A = \{\mathbf{a}+\mathbf{e}_i : i \in F\}$ for some fixed vector $\mathbf{a} = (a_1, \ldots, a_n)$, such that $a_1+\cdots+a_n = r-1$ and $a_i \geq 0$.

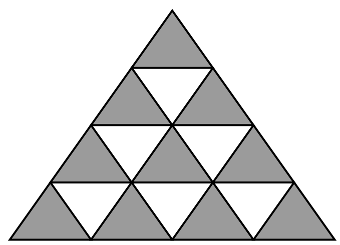


Figure 3.2: The 4-fold half-hollow edgewise subdivision of a 2-simplex. If we "fill in" the holes, then we obtain the original edgewise subdivision introduced by Edelsbrunner and Grayson.

Example 3.7.2 (The hesd **construction for graphs).** Let $r > 0$ and G be a graph with vertex set $[n]$ and edge set E. Condition (HESD1) implies edges of $\operatorname{hesd}(G, r)$ are of the form $\mathbf{a} = a_i\mathbf{e}_i + a_j\mathbf{e}_j, \mathbf{b} = b_i\mathbf{e}_i + b_j\mathbf{e}_j$ for $\{i, j\} \in E$. Condition (HESD2) then implies $\operatorname{hesd}(G, r)$ is obtained from G by replacing every edge $\{i, j\}$ of G by a path on $r + 1$ vertices with end points i, j.

Our next goal is to show that given a d-dimensional complex Δ, the log-matrix of $I(\operatorname{hesd}(\Delta(d), a-1))$ represents multiplication maps in algebras of the form $A_\Delta(a)$.

Theorem 3.7.3. *Let Δ be a d-dimensional pure simplicial complex, $a > 1$ an integer, $L = x_1+\cdots+x_n \in R = \mathbb{K}[x_1, \ldots, x_n]$ and $J = I_\Delta+(x_1^a, \ldots, x_n^a)$. Then the matrix representing the map $\times L : (R/J)_{d(a-1)} \to (R/J)_{d(a-1)+1}$ is the log-matrix of $I(\operatorname{hesd}(\Delta(d), a-1))$.*

Proof. Let $A_\Delta(a) = R/J$, $t = d(a-1)$ and M be the matrix representing the map $\times L : A_\Delta(a)_t \to A_\Delta(a)_{t+1}$. Note that nonzero monomials of $A_\Delta(a)_{t+1}$ are of the form $m = x_{i_1}^{b_1} \ldots x_{i_{d+1}}^{b_{d+1}}$, where $\{i_1, \ldots, i_{d+1}\} \in \Delta$, $b_1 + \cdots + b_{d+1} = t+1$ and $b_j < a$ for all j. In particular, both conditions together imply $b_j > 0$ for every j. We then conclude $L \circ m$

is a sum of $d+1$ monomials of degree t for every nonzero monomial $m \in A_\Delta(a)_{t+1}$. In other words, the row of M corresponding to the monomial m has $d+1$ nonzero entries, and they are all 1. This implies M is the log-matrix of the facet ideal of a pure complex Γ of dimension d.

Our next goal is to show that $\Gamma = \mathrm{hesd}(\Delta(d), a-1)$. Note that vertices of Γ correspond to columns of M and hence are labeled by nonzero monomials in $A_\Delta(a)_t$, and the facets of Γ correspond to nonzero monomials in $A_\Delta(a)_{t+1}$.

In order to show the equality $\Gamma = \mathrm{hesd}(\Delta(d), a-1)$, we first need to show a correspondence between vertices of Γ and vertices of $\mathrm{hesd}(\Delta(d), a-1)$ which by definition is $\Omega^{a-1}_{f_{d-1}} \subset \mathbb{R}^{f_{d-1}}$. For a facet F of Δ, set $\omega_F = \prod_{i\in F} x_i^{a-1}$. For every nonzero monomial $m \in A_\Delta(a)_t$, there exists a facet F such that we can write m as $m = \frac{\omega_F}{x_{j_1}^{c_1}\dots x_{j_s}^{c_s}}$, where $c_1 + \cdots + c_s = a-1$. Now, there is a bijection between ridges of F and vertices of F given by $v \mapsto F \setminus v$. Since ridges of F are also ridges of Δ, we define a function from the vertices of Γ (which are monomials in $A_\Delta(a)_t$) to the vertices of $\mathrm{hesd}(\Delta(d), a-1)$ given by

$$\varphi : V(\Gamma) \to \Omega^{a-1}_{f_{d-1}}, \quad \varphi\Big(\frac{\omega_F}{x_{j_1}^{c_1}\dots x_{j_s}^{c_s}}\Big) = c_1\mathbf{e}_{F\setminus j_1} + \cdots + c_s\mathbf{e}_{F\setminus j_s},$$

where $\mathbf{e}_\tau$ is the element in the canonical basis of $\mathbb{R}^{f_{d-1}}$ corresponding to the ridge τ of Δ. We note that even though it is possible for a monomial m to have two different representations $\frac{\omega_{F'}}{x_{j_1}^{c_1}\dots x_{j_s}^{c_s}} = \frac{\omega_{F''}}{x_{k_1}^{q_1}\dots x_{k_\ell}^{q_\ell}} = m \in A_\Delta(a)_t$, since the degree of the denominator is $a-1$ and the support of m is a face of Δ, this only happens if $m = \prod_{i\in\tau} x_i^{a-1}$ for some ridge $\tau = F' \setminus v' = F'' \setminus v''$ of Δ. In this case, $\varphi(m) = (a-1)\mathbf{e}_\tau$ and so φ is well defined. The map φ is a surjection since for every vector $\mathbf{a} = c_1\mathbf{e}_{F\setminus j_1} + \cdots + c_s\mathbf{e}_{F\setminus j_s} \in \Omega^{a-1}_{f_{d-1}}$, the monomial $m = \frac{\omega}{x_{j_1}^{c_1}\dots x_{j_s}^{c_s}} \in A_\Delta(a)_t$ is such that $\varphi(m) = \mathbf{a}$. To see that φ is an injection, note that if $\varphi(m) = \varphi(m') = \mathbf{a} = c_1\mathbf{e}_{F\setminus j_1} + \cdots + c_s\mathbf{e}_{F\setminus F_{j_s}}$, then we have $\mathrm{supp}(m) = \mathrm{supp}(m')$ and in particular $m = \frac{\omega_F}{x_{j_1}^{c_1}\dots x_{j_s}^{c_s}} = m'$.

For a correspondence between facets of Γ (which are monomials in $A_\Delta(a)_{t+1}$) and facets of $\mathrm{hesd}(\Delta(d), a-1)$, note that every facet of Γ is given by the set of monomials $\{\frac{m}{x_i} \colon x_i \mid m\}$, where $m \in A_\Delta(a)_{t+1}$. Similarly to the case of monomials of degree t described above, every monomial m of degree $t+1$ can be written as $m = \frac{\omega_F}{x_{j_1}^{c_1}\dots x_{j_s}^{c_s}}$ for some facet F of Δ, where $c_1 + \cdots + c_s = a-2$. Denoting by $\mathbf{a}_m$ the vector associated to the monomial m of degree $t+1$ we see that facets of Γ correspond to sets of the form $\{\mathbf{a}_m + \mathbf{e}_{F\setminus i} \colon i \in F\}$, and since $\mathbf{a}_m$ is a vector with positive entries adding up to $a-2$,

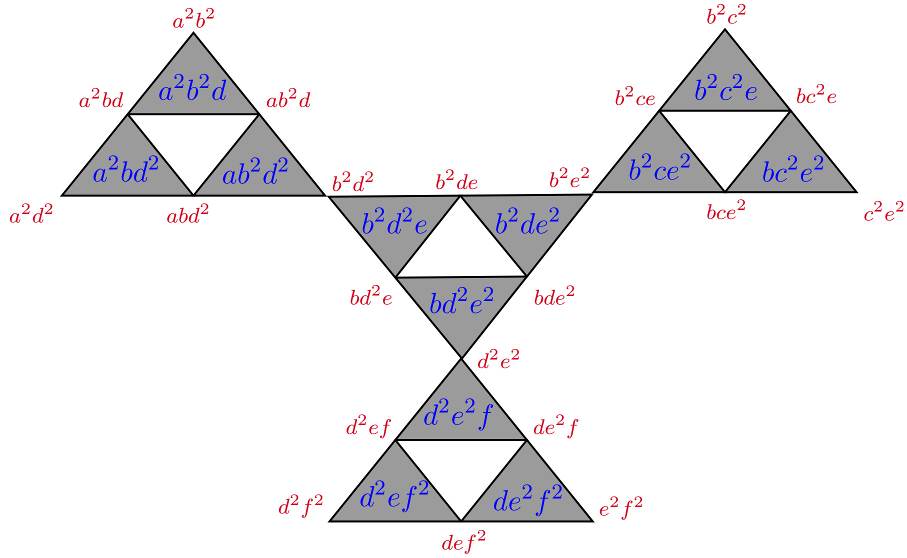


Figure 3.3: A labeling of the vertices and facets of $\mathrm{hesd}(\Delta(2), 2)$, where Δ is the complex from Figure 3.1 and $I(\Delta) = (abd, bde, bce, def)$, that shows the log-matrix of $I(\mathrm{hesd}(\Delta(2), 2))$ is the multiplication map $\times L : A_\Delta(3)_4 \to A_\Delta(3)_5$. Note that applying $\times L^T$ to the monomial label of a triangle yields the sum of the labels of its vertices. As an example, the triangle labeled by a^2b^2d corresponds to the vector $\mathbf{a} = \mathbf{e}_{\{a,b\}}$ and its vertices labeled by a^2b^2, a^2bd and ab^2d correspond to the vectors $2\mathbf{e}_{\{a,b\}}$, $\mathbf{e}_{\{a,b\}} + \mathbf{e}_{\{a,d\}}$ and $\mathbf{e}_{\{a,b\}} + \mathbf{e}_{\{b,d\}}$.

and $\mathrm{supp}(a)$ by definition is the set of ridges of a facet of Δ, we conclude facets of Γ correspond exactly to the sets of vectors satisfying (HESD1), (HESD2).

We have shown that there is a bijection between vertices of Γ and $\mathrm{hesd}(\Delta(d), a-1)$ that preserves facets, in particular both complexes have the same log matrices and so the result follows. $\square$

3.7.2 Collapsibility and upper triangular matrices

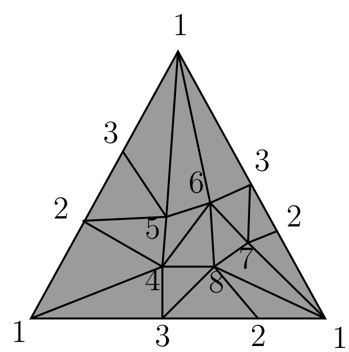
In Section 3.7.1 we introduced two constructions: incidence complexes and the half-hollow edgewise subdivision of a simplicial complex Δ. These constructions led us to Theorem 3.7.3, but having an equality of matrices alone is not enough to conclude any information on the existence (or nonexistence) of unexpected s.o.p.s. We now recall the topological notion of *collapsibility*, which will allow us to use the results in Section 3.7.1 to guarantee the *nonexistence* of unexpected s.o.p.s in certain settings.

A face σ of a simplicial complex Δ is *free* if it is properly contained by a unique face τ of Δ.

Definition 3.7.4 (Elementary collapses and collapsible complexes). Let Δ be a simplicial complex. An *elementary collapse* of Δ by a free face σ, is the process of deleting the faces σ, τ from Δ, resulting in a new complex Δ'. In this case we say Δ' is obtained from Δ via an elementary collapse. A complex Δ is said to be *collapsible* if the complex with a single vertex can be obtained from Δ via a sequence of elementary collapses.

Collapsibility was introduced by Whitehead in 1939 [203] and since then it has become one of the most well-studied concepts in combinatorial topology. One of the main reasons collapsible complexes, and more generally, elementary collapses, have become such a useful tool, comes from the fact that if Γ is obtained from Δ via an elementary collapse, then Γ and Δ are homotopy equivalent. In particular, collapsible complexes are contractible. The converse, however, is far from true.

Example 3.7.5 (Collapsible vs contractible complexes [206]). The complex Δ below is a triangulation of the *Dunce hat*, one of the earliest examples of a contractible but non collapsible complex.



The property of collapsible complexes that is relevant to us is the following.

Proposition 3.7.6. *Let Δ be a d-dimensional pure collapsible simplicial complex and $r > 0$. Then the log-matrix of $I(\operatorname{hesd}(\Delta(d), r))$ has full rank. In particular, $I(\operatorname{hesd}(\Delta(d), r))$ has maximal analytic spread.*

Proof. By Theorem 2.7.5 and since $\operatorname{hesd}(\Delta(d), r)$ is pure, the analytic spread of $I(\operatorname{hesd}(\Delta(d), r))$ is equal to the rank of the log-matrix M of $I(\operatorname{hesd}(\Delta(d), r))$. We will show that under the assumptions of the theorem, this matrix is upper triangular with nonzero diagonal entries, and hence must have full rank.

We begin by showing that for a simplex Γ on n vertices, the log-matrix N of the facet ideal of the complex $\operatorname{hesd}(\Gamma, r) = \langle F_1, \ldots, F_s \rangle$ is upper triangular. In order to show this, we need to find an injection ψ from the facets of $\operatorname{hesd}(\Gamma, r)$ to the vertices of $\operatorname{hesd}(\Gamma, r)$ such that (after reordering the facets of $\operatorname{hesd}(\Gamma, r)$) we have $\psi(F_i) = v_i \notin F_j$ for $j > i$. Since by (HESD2) the set of facets of $\operatorname{hesd}(\Gamma, r)$ is Ω_n^{r-1} and the set of vertices is Ω_n^r, we may define our injection $\psi = \psi_1$ as

$$\psi_1(\mathbf{a}) = \mathbf{a} + \mathbf{e}_1, \tag{3.12}$$

where the ordering of facets of $\operatorname{hesd}(\Gamma, r)$ is the lex order on the elements of Ω_n^{r-1}, in other words, $\mathbf{a} <_{\text{lex}} \mathbf{b}$ if and only if there exists t such that $a_i = b_i$ for all $i < t$ and $a_t < b_t$. Under these assumptions, $\psi(\mathbf{a})$ is a vertex of a facet corresponding to $\mathbf{a}'$ if and only if $\mathbf{a}' = \mathbf{a} + \mathbf{e}_j$ for some j. As the facets are ordered lexicographically (as elements of Ω_n^{r-1}), if $\mathbf{a} + \mathbf{e}_1 = \mathbf{b} + \mathbf{e}_j$ for some $j > 1$, we conclude $\mathbf{a} <_{\text{lex}} \mathbf{b}$, in other words, ψ satisfies the desired property.

Now for an arbitrary d-dimensional pure collapsible complex Δ, let $\sigma_1, \ldots, \sigma_s$ be a sequence of $(d-1)$–faces such that after performing elementary collapses using the σ_i, the resulting complex is a $(d-1)$-dimensional complex. To see that the log-matrix M of $I(\operatorname{hesd}(\Delta(d), r))$ is upper triangular (after reordering rows and columns), for a facet T of $\Delta(d)$, let

$$C_T = \{\tau \colon \cup_{\mathbf{a} \in \tau} \operatorname{supp}(\mathbf{a}) = T \text{ and } \tau \text{ a facet of } \operatorname{hesd}(\Delta(d), r)\}.$$

Then the previous arguments show that for T_1, the unique facet of Δ containing σ_1, there exists an injection ψ_1 from facets in C_{T_1}, to vertices of $\operatorname{hesd}(\Delta(d), r)$, where we reorder the vertices to guarantee that $\mathbf{e}_1$ is the coordinate corresponding to σ_1. After deleting every face of $\operatorname{hesd}(\Delta(d), r)$ that is contained only in facets of C_{T_1}, we obtain the complex $\operatorname{hesd}(\Delta'(d), r)$, where Δ' is the complex obtained after the first elementary collapse involving σ_1. We may now repeat the argument for the unique facet T_2 of Δ' that contains σ_2. Inductively, since the ridges $\sigma_1, \ldots, \sigma_s$ were chosen such that the sequence of elementary collapses using them yields a $(d-1)$–dimensional complex, we conclude the matrix is upper triangular. $\square$

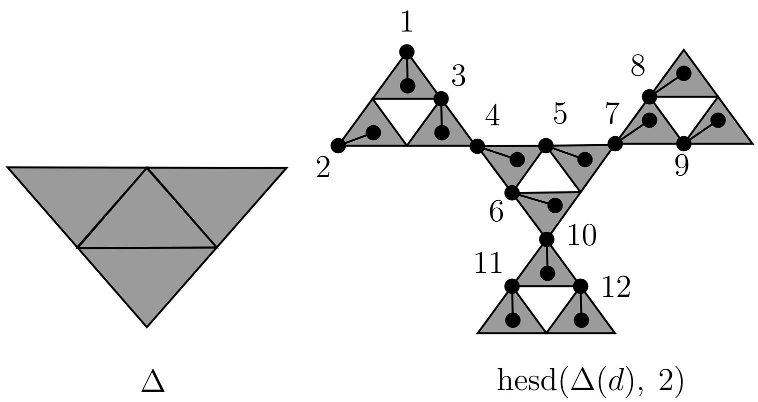


Figure 3.4: The complex Δ on the left, and the complex $\mathrm{hesd}(\Delta(2), 2)$ on the right. The lines from vertices to facets correspond to the injection from Proposition 3.7.6 that shows the log matrix of $I(\mathrm{hesd}(\Delta(2), 2))$ is upper triangular. Note that for every line, the vertex i is contained only in facets corresponding to lower values of i, for example, the vertex 3 is also contained in the facet containing the vertex 1.

We are now ready to state the main result of this chapter: a setting where we can guarantee that unexpected s.o.p.s do not exist.

Corollary 3.7.7 (A setting where unexpected sops do not exist). *Let Δ be a d-dimensional Cohen-Macaulay collapsible complex. Then for a fixed $a > 0$, there does not exist an a-unexpected system of parameters of Δ of total degree $t > d(a-1)$.*

Proof. Let $L = x_1 + \cdots + x_n \in R$. By Theorems 2.7.5 and 3.7.3 we have

$$\mathrm{rank} \times L : A_\Delta(a)_{d(a-1)} \to A_\Delta(a)_{d(a-1)+1} = \ell(I(\mathrm{hesd}(\Delta(d), a-1))).$$

Proposition 3.7.6 then implies the map is surjective.

If $\theta_1, \ldots, \theta_{d+1}$ is an a-unexpected sop of Δ of total degree t, the socle degree of the algebra $\frac{R}{I_\Delta + (\theta_1, \ldots, \theta_{d+1})}$ is t and so by Proposition 3.4.6 a top degree generator F in the inverse system of $I_\Delta + (\theta_1, \ldots, \theta_{d+1})$ corresponds to a nonzero element in the kernel of $\times L^\top : (R/J)_t \to (R/J)_{t-1}$, which in turn implies the map $\times L : (R/J)_{t-1} \to (R/J)_t$ is not surjective. This is a contradiction since once a map $\times L : (R/J)_i \to (R/J)_{i+1}$ is surjective, every multiplication map $\times L : (R/J)_j \to (R/J)_{j+1}$ for $j \geq i$ is also surjective (see for example Proposition 2.5.12). $\square$

Remark 3.7.8 (A weaker property than collapsing.). It should be noted that every result we proved that assumed Δ is a collapsible complex still holds if we only assume Δ collapses down to a lower dimensional complex. We state results in the language of collapsibility for the sake of simplicity.

Remark 3.7.9 (Why analytic spread?). In Proposition 3.7.6 we stated the result (and proved it) using the language of analytic spread. In view of Theorem 2.7.5, this is equivalent (in our setting) to saying a matrix has full rank. The advantage of including the analytic spread language, is that it allows for the connection between different areas of commutative algebra such as Lefschetz properties and Rees algebras, to become more apparent.

In [92, 95] we explored these connections, and showed many connections between properties related to Rees algebras such as: positivity of mixed multiplicities, linear type and being normally torsion-free, and an algebra either having or failing the WLP/SLP.

Chapter 4

Gluing failures to achieve success: Simplicial complete intersections and probabilistic failure

> *"Esses fatos parecerão a alguns perfeitamente naturais e a outros, ao contrário, inverossímeis. Mas, afinal, um cronista não pode levar em conta essas contradições. Sua tarefa é apenas dizer 'Isso aconteceu', quando sabe que isso, na verdade, aconteceu"*
>
> ALBERT CAMUS – *A Peste*

In Chapter 3 we introduced the notion of an *unexpected s.o.p.* and showed that if Δ admits an unexpected s.o.p. then a certain monomial ideal containing I_Δ fails the weak Lefschetz property (WLP). The word "unexpected" comes from the geometric theory of unexpected hypersurfaces. As was mentioned in Section 3.3, in general, given an artinian algebra A, failure of the WLP is often seen as the consequence of some interesting phenomenon. In particular, artinian algebras are often expected to satisfy the WLP/SLP. This belief is in fact justfied, as Watanabe [199, Example 3.9] pointed out in 1987: "In view of Remark 3.7, (2), it turns out that 'most' Gorenstein rings have *SSP*."[1]

The main goal of this chapter is to show that there are settings where the WLP of artinian algebras *should not be expected*. More concretely, we will develop techniques that suggest algebras of the form

$$A_\Delta(a) = \frac{R}{I_\Delta + (x_1^a, \dots, x_n^a)} \quad \text{where} \quad R = \mathbb{K}[x_1, \dots, x_n]$$

should be expected to *fail* the WLP.

It should be noted that in algebraic settings, the word "most" (such as in Watanabe's quote above) usually means some property holds in a Zariski-open set. For us, since the algebras $A_\Delta(a)$ fit more naturally in a combinatorial setting, the word "most" will mean

[1] At the time, the SLP was called the *strong Stanley property (SSP), instead of strong Lefschetz property.*

some property holds (or fails in our case) with high probability. As a consequence, in order to even formalize the idea behind our statements, we need to introduce a probability space for the algebras $A_\Delta(a)$ above (or equivalently, for simplicial complexes Δ). The probability space we consider is a well-known space from the field of random topology, and its definition will be recalled in Section 4.2

The ideas and results from this chapter are essentially those in [93], but we will provide statements here in slightly different settings, with the hope of showing that there is a lot more that can be done with similar techniques.

4.1 An algebraic motivation: almost complete intersections

One of the biggest problems in the study of Lefschetz properties is the following[2].

Question 4.1.1 ([170, pp. 534] and [146, Question 3.1])**.** *Does every artinian complete intersection over a field of characteristic zero satisfy the WLP/SLP?*

Question 4.1.1 has been extensively studied throughout the years and has been one of the main motivations for many of the strongest results in the theory of Lefschetz properties of artinian algebras. Two examples of related statements are the following. A proof of both statements can be found in [79].

1. Every artinian quotient of $\mathbb{K}[x, y]$ satisfies the SLP.

2. Every artinian complete intersection that is a quotient of $\mathbb{K}[x, y, z]$ where $\mathbb{K}$ is a field of characteristic zero satisfies the WLP.

Question 4.1.1 asks whether every artinian ideal $I \in \mathbb{K}[x_1, \dots, x_n]$ generated by n elements satisfies the WLP, where $\mathbb{K}$ is a field of characteristic zero. In an attempt to generalize Question 4.1.1, Migliore and Miró-Roig asked the following question in 2003.[3]

Question 4.1.2 ([144, Question 4.2])**.** *For any integer $n \geq 3$, find the maximum number $A(n)$ (if it exists) such that every artinian ideal $I \subset \mathbb{K}[x_1, \dots, x_n]$ generated by at most*

[2]There is no consensus between experts on whether the answer to Question 4.1.1 should be positive or negative. The question was originally stated by Reid, Roberts and Roitman in 1991 [170], where they conjectured that the answer to the question is positive.

[3]Right after stating Question 4.1.2, Migliore and Miró-Roig ask whether $A(n) = 2n - 2$ for every $n \geq 3$. The bound $2n - 2$ comes from a specific example the authors found in [144, Example 4.1].

$A(n)$ elements defines an algebra with the WLP. Included in this question is whether every complete intersection in $\mathbb{K}[x_1, \ldots, x_n]$ has the WLP, which would say that $A(n) \geq n$.

In view of one of the results mentioned above, we conclude $A(3)$ exists, and in particular $A(3) \geq 3$. Shortly after in 2007, Brenner and Kaid [20] used techniques from algebraic geometry in order to study the WLP and as a corollary showed that the algebra $\mathbb{K}[x, y, z]/(xyz, x^3, y^3, z^3)$ fails the WLP over every $\mathbb{K}$. Brenner and Kaid's example [20, Example 3.1] was one of the first interesting examples of a level monomial algebra failing the WLP, and it showed that $A(3) = 3$. Motivated by Brenner and Kaid's results, Migliore, Miró-Roig and Nagel [148] in 2011 began the study of *monomial almost complete intersections*, that is, the WLP of algebras defined by powers of the variables plus one extra monomial generator. One of the main results in [148] is the following result.[4]

Theorem 4.1.3 (**Monomial almost complete intersections and the WLP,** [148, Theorem 4.3]**).** *For any n and any field $\mathbb{K}$, the algebra*

$$\frac{\mathbb{K}[x_1, \ldots, x_n]}{(x_1 \ldots x_n) + (x_1^n, \ldots, x_n^n)}$$

fails the WLP.

Theorem 4.1.3 led to the question of which monomial almost complete intersections define an algebra satisfying the WLP. More generally, [148] is the starting point of the study of Lefschetz properties of algebras defined by monomial ideals.

It should be noted that surprisingly, even though the study of the WLP of monomial ideals has been shown to be related to many areas of combinatorics, such as the enumeration of combinatorial objects (see for example [33]), it was only in 2020 that Migliore, Nagel and Schenck [147] used Stanley-Reisner theory to study the WLP of algebras of the form

$$\frac{\mathbb{K}[x_1, \ldots, x_n]}{I(G) + (x_1^2, \ldots, x_n^2)} \quad \text{for some} \quad \text{graph } G.$$

The point of the results in [147] is that when $\mathbb{K}$ is a field of characteristic 2, the multiplication maps by $x_1 + \cdots + x_n$ are equal to the boundary maps (over $\mathbb{K}$) of the independence

[4]Considering the context of the questions being asked in the early 2000's (such as [144], where Migliore and Miró-Roig asked whether $A(n) = 2n-2$ for all $n \geq 3$), Theorem 4.1.3 is a shocking result: not only does it show that the number $A(n)$ (if it exists) must be equal to n for every n, but it shows "simple" monomial ideals can fail the WLP.

complex of G.

Since [147], most of the applications have focused on combinatorial aspects of simplicial complexes and graphs. The results in this chapter do not follow the same trend, and instead, we take a more topological point of view.

4.2 A topological theory: random complexes

Before starting this section, we note that we will need standard definitions from probability theory such as the notion of probability space, (Bernoulli and binomial) distributions and independence. We refer the reader to [91] for definitions.

At the 2023 Lefschetz meeting at the Fields Institute in Toronto, Nagel proposed the following broad question:

Question 4.2.1. *When should the WLP of artinian algebras defined by monomial ideals be expected? More specifically, given a probability distribution with parameter c in the set of all artinian monomial ideals, for which values of c does a random artinian monomial algebra satisfy the WLP with high probability?*

It is clear that an answer to Question 4.2.1 depends heavily on the chosen probability distribution. Here we will focus solely on monomial algebras of the form

$$A_\Delta(a) = \frac{\mathbb{K}[x_1, \ldots, x_n]}{I_\Delta + (x_1^a, \ldots, x_n^a)}.$$

In particular, since for fixed a there is a bijection between algebras we consider and simplicial complexes on n vertices, we only need to consider probability distributions on the set of all such simplicial complexes. The probability distributions we will consider in this chapter have been extensively studied from combinatorial and topological perspectives. We begin by introducing the basics of Erdős-Rényi theory.

In 1959, Erdős and Rényi [50] began the study of *random graphs*, which consists of studying graph theoretical invariants (and properties) of graphs obtained from a random process. Let $\mathcal{Y}_1(n)$ be the set of all graphs on n vertices. We turn $\mathcal{Y}_1(n)$ into a probability space $\mathcal{Y}_1(n,p)$ with parameter p by defining a probability distribution[5] such that

$$\mathbb{P}(Y \in \mathcal{Y}_1(n,p)) = p^{f_1}(1-p)^{\binom{n}{2}-f_1} \quad \text{where} \quad f(Y) = (1, n, f_1)$$

is the f-vector of Y.

The probability distribution has a simple interpretation: the graph Y is obtained by making a (biased) coin flip for every pair of integers in $[n]$, if the result is heads (which happens with probability p), the pair is added to the graph as an edge. If the output is tails (which happens with probability $1-p$), we move on to the next pair. More formally, denote by $\binom{[n]}{d+1}$ the set of all subsets of $[n]$ of size $d+1$. For every $A = \{i,j\} \in \binom{[n]}{2}$, let X_A be a Bernoulli random variable with probability parameter p and assume all X_A are independent. Then a graph can be thought of as a a specific output of the vector of random variables

$$(X_A \colon A \in \binom{[n]}{2}).$$

Two of the most fundamental theorems about random graphs (see [51]) describe values of c for which the following statements hold:

1. $\lim_{n\to\infty} \mathbb{P}(Y \in \mathcal{Y}_1(n, \frac{c}{n}) \colon Y$ is connected$) = 1$, and
2. $\lim_{n\to\infty} \mathbb{P}(Y \in \mathcal{Y}_1(n, \frac{c}{n}) \colon Y$ has a cycle$) = 1$.

At this point, notice that everything we have discussed so far in this section, from definitions to statements, can be generalized from graphs to higher dimensional complexes. As examples, statements about connectivity of graphs can be thought of as a d-dimensional complex having trivial $(d-1)$–homology, similarly, statements about graphs containing cycles can be thought of as a d-dimensional complex having nontrivial top homology.

In view of the remarks above, it is natural to consider the probability space $\mathcal{Y}_d(n,p)$ on the set $\mathcal{Y}_d(n)$ of all d-dimensional complexes with $f_{d-1} = \binom{n}{d}$, and probability distribution

$$\mathbb{P}(Y \in \mathcal{Y}_d(n,p)) = p^{f_d}(1-p)^{\binom{n}{d+1}-f_d} \quad \text{where} \quad f(Y) = (1, n, \dots, \binom{n}{d}, f_d).$$

The first ones to consider this generalization were Linial and Meshulam [121] who introduced the model for $d = 2$ in 2006. Then, in 2009, Meshulam and Wallach [141]

[5]There is more than one model for random graphs, and many models are related. Here, we will only need the one defined, and the connections will not be relevant.

considered the full generalization for arbitrary d. The results in [121, 141] mostly tried to find thresholds for p such that the $(d-1)$–homology has a high probability of being 0. The result from this theory we will need is due to Aronshtam, Linial, Łuczak and Meshulam [8], which generalizes the statement about cycles in random graphs mentioned above.

Before stating the result, we first need to introduce some notation from [8]. Set $c_1 = 1$. For every $d \geq 2$, set

$$g_d(x) = (d+1)(x+1)e^{-x} + x(1-e^{-x})^{d+1},$$

and c_d the unique[6] positive solution to $g_d(x) = d+1$. It can be shown (see [8, pp. 318]) that $c_d < d+1$ for every d (for $d = 1$ it follows directly by definition), and $\lim_{d\to\infty} c_d = d+1$. The following statement was shown (for $d > 1$) in [8], and strenghthened an earlier result due to Koszlov [114]. The case $d = 1$ is a well-known result, and it says the probability of a random graph on n vertices containing a cycle converges to 1 as n goes to infinity, whenever $p > 1/n$ (see the original paper [51]).[7]

Theorem 4.2.2 ([8, Theorem 1.2], [51])**.** *For a fixed $c > c_d$ and an arbitrary field $\mathbb{K}$ we have:*

$$\lim_{n\to\infty} \mathbb{P}(Y \in \mathcal{Y}_d(n, \frac{c}{n})\colon \tilde{H}_d(Y;\mathbb{K}) \neq 0) = 1.$$

We are going to to use Theorem 4.2.2 by showing that nontrivial homology in a simplicial complex implies the failure of surjectivity of a certain multiplication map in an artinian algebra. Another result we need from the model is the following, which will guarantee that the same multiplication map that fails to be surjective, must also fail to have full rank.

Lemma 4.2.3 ([93, Theorem 7.4])**.** *Let $d \geq 1$, $c < d+1$. Then*

$$\lim_{n\to\infty} \mathbb{P}(Y \in \mathcal{Y}_d(n, \frac{c}{n})\colon f_d \leq f_{d-1}) = 1.$$

Proof. Note that f_d counts the number of successes in the Bernoulli variables X_A. Since the Bernoulli variables are all independent and have the same probability parameter p, we conclude $f_d = \sum_{A\in\binom{[n]}{d+1}} X_A$ is a binomial random variable with parameters $N = \binom{n}{d+1}$

[6]The fact that $g_d(x) = d+1$ has a unique positive solution can be seen from a careful study of the derivative of $g_d(x)$.

[7]In [8] the authors assume $d \geq 2$ throughout the whole paper, since many of their results were either already known, or do not apply to the $d = 1$ case.

and p is the same as the probability parameters of the Bernoulli random variables. In particular, we know the expected value of f_d is

$$pN = \frac{c}{n}\binom{n}{d+1}.$$

Set $\delta = \frac{d+1}{c}\frac{n}{n-d} - 1$ and note that

$$(1+\delta)\binom{n}{d+1}\frac{c}{n} = \frac{d+1}{c}\frac{n}{n-d}\frac{c}{n}\binom{n}{d+1} = \frac{d+1}{n-d}\binom{n}{d+1} = \binom{n}{d},$$

so we conclude

$$\mathbb{P}(Y \in \mathcal{Y}_d(n, \frac{c}{n})\colon f_d \geq f_{d-1}) = \mathbb{P}(f_d \geq f_{d-1}) = \mathbb{P}(f_d \geq \binom{n}{d}) = \mathbb{P}(f_d \geq (1+\delta)\binom{n}{d+1}).$$

Since $c < d+1$, we conclude $\delta > 0$. Chernoff's bound [151, Theorem 4.4] then implies:

$$\mathbb{P}(f_d \geq (1+\delta)\binom{n}{d+1}) \leq e^{-\frac{\min(\delta,\delta^2)}{4}\frac{c}{n}\binom{n}{d+1}}.$$

As $d, \delta > 0$ and $\binom{n}{d+1}$ is a polynomial in n of degree $d+1$, a calculus argument implies

$$1 = 1 - \lim_{n\to\infty} e^{-\frac{\min(\delta,\delta^2)}{4}\frac{c}{n}\binom{n}{d+1}} \leq \lim_{n\to\infty} \mathbb{P}(f_d < f_{d-1}) \leq \lim_{n\to\infty} \mathbb{P}(f_d \leq f_{d-1}) \leq 1.$$

$\square$

Combining Theorem 4.2.2 and Lemma 4.2.3, together with the fact that $c_d < d+1$, we have the following statement[8] which we will apply in future sections:

Lemma 4.2.4 ([93, Theorem 7.4])**.** *Let $\mathbb{K}$ be an arbitrary field, $d \geq 1$ and $c_d < c < d+1$. Then*

$$\lim_{n\to\infty} \mathbb{P}(Y \in \mathcal{Y}_d(n, \frac{c}{n})\colon f_d \leq f_{d-1} \quad \textit{and} \quad \tilde{H}_d(Y;\mathbb{K}) \neq 0) = 1.$$

Proof. The probability in the statement is of the form $\lim_{n\to\infty} \mathbb{P}(E_1(n) \cap E_2(n))$ where, by Theorem 4.2.2 and Lemma 4.2.3, we know

$$\lim_{n\to\infty} \mathbb{P}(E_1(n)) = \lim_{n\to\infty} \mathbb{P}(E_2(n)) = 1.$$

[8] In [93] both statements appear in the proof of a theorem that we will state in a later section (see Theorem 4.4.6).

The result then follows by the formula

$$\lim_{n\to\infty} \mathbb{P}(E_1(n) \cap E_2(n)) = \lim_{n\to\infty} \mathbb{P}(E_1(n)) + \lim_{n\to\infty} \mathbb{P}(E_1(n)) - \lim_{n\to\infty} \mathbb{P}(E_1(n) \cup E_2(n)),$$

and since $\lim_{n\to\infty} \mathbb{P}(E_1(n) \cup E_2(n)) = 1$. □

4.3 A combinatorial insight: anti-symmetric polynomials

The most naive strategy for proving failure of WLP for an artinian monomial algebra A, is to show that for some i:

Step 1 $\times L : A_i \to A_{i+1}$ is not surjective, where L is the sum of variables, and

Step 2 $\mathrm{HF}_A(i) \geq \mathrm{HF}_A(i+1)$.

While the strategy seems pretty simple, both steps are almost always very hard to prove (for large families of algebras). This strategy was in fact the strategy used in [93, 148]. With this strategy in mind, we can put the results from Section 4.2 in the context of Lefschetz properties. Our goal is to show that Step 1 follows from the nonvanishing of the top homology of a simplicial complex, via an application of Theorem 4.2.2, and Step 2 follows from a certain inequality on the f-vector of a simplicial complex, via an application of Lemma 4.2.3. In this section, we will focus specifically on Step 1, and our results will lead us to the second main object of this thesis: *simplicial complete intersections*, which are special examples of unexpected systems of parameters.

4.3.1 (Homological) Extensions of anti-symmetric polynomials

A key step[9] in the proof of Theorem 4.1.3 is showing that the *Vandermonde determinant* $V = \prod_{1\leq i<j\leq n}(x_i - x_j) \in R = \mathbb{K}[x_1, \ldots, x_n]$ is in the kernel of the map

$$\times L^\top : \left(\frac{R}{(x_1 \ldots x_n, x_1^n, \ldots, x_n^n)}\right)_{\binom{n}{2}} \to \left(\frac{R}{(x_1 \ldots x_n, x_1^n, \ldots, x_n^n)}\right)_{\binom{n}{2}-1} \quad \text{where} \quad L = \sum_{i=1}^n x_i,$$

[9] Note that in this case finding the polynomial is perhaps harder than proving the polynomial is in the kernel.

and hence the multiplication from degree $\binom{n}{2} - 1$ to $\binom{n}{2}$ is not surjective. From the perspective of our results, this statement can be proven almost directly by Lemma 3.4.3 and since V is the Macaulay dual generator of $(L, e_2, \ldots, e_n)$.

Another (less famous) property of the Vandermonde determinant, is that it can be broken down into smaller Vandermonde determinants. Let $m = x_1 \ldots x_n$, and write $V(A)$ for the Vandermonde determinant using variables in A, that is

$$V(A) = \prod_{i,j \in A,\ i<j} (x_i - x_j).$$

Then we can write the Vandermonde determinant as

$$V([n]) = (-1)^1 \frac{m}{x_1} V([n] \setminus 1) + \cdots + (-1)^n \frac{m}{x_n} V([n] \setminus n). \tag{4.1}$$

More concretely, we can rewrite the Vandermonde polynomial as a sum of smaller Vandermondes, with coefficients of ± 1, and the product of all the variables appearing in the smaller Vandermonde. At first this might seem unnecessary, but by changing our perspective a little, it will become an important tool for proving results in this chapter. The two properties relevant to us are:

1. (gluing faces) Every term in the decomposition in (4.1) uses only variables in each facet of the boundary of the simplex on $[n]$ vertices.

2. (over homology cycles) The coefficients of each term are equal to the coefficients of the orientation of the boundary of the simplex.

These two properties suggest that a topological perspective could be useful. Our next goal is formalize this idea.

Recall that a polynomial $f(x_1, \ldots, x_n)$ is called *anti-symmetric* if

$$f(x_{\pi(1)}, \ldots, x_{\pi(n)}) = \operatorname{sgn}(\pi) f(x_1, \ldots, x_n)$$

for every permutation π of $[n]$ elements, where $\operatorname{sgn}(\pi)$ is the sign of the permutation.

Set $d > 0$. Let Δ be a d-dimensional simplicial complex on vertex set $[n]$ and $C_\bullet$ its chain complex over an arbitrary field $\mathbb{K}$. For every d-facet σ of Δ, let $R_\sigma = \mathbb{K}[x_i \colon i \in \sigma]$. Set $B = \mathbb{K}[z_1, \ldots, z_{d+1}]$ and let $f \in B$ be a homogeneous anti-symmetric polynomial.

Given that the vertices of Δ are ordered (i.e. an orientation of the vertices is fixed), for every d-facet σ of Δ there is a natural map $\varphi_\sigma : B \to R_\sigma$ such that $\varphi(z_i) = x_{j_i}$, where $\sigma = \{j_1, \ldots, j_{d+1}\}$ and $j_1 < \cdots < j_{d+1}$. We will always assume $\tilde{H}_d(\Delta; \mathbb{K}) \neq 0$ and denote by ε a nonzero element of $\tilde{H}_d(\Delta; \mathbb{K})$. Moreover, we write $\varepsilon(\sigma)$ for the coefficient of σ in the cycle ε.

Definition 4.3.1 (Homological extensions of anti-symmetric polynomial). Following notation above, the *ε-extension* of the anti-symmetric polynomial $f \in B$ is the polynomial

$$F_\varepsilon = \varepsilon(\sigma_1) \prod_{i \in \sigma_1} x_i \varphi_{\sigma_1}(f) + \cdots + \varepsilon(\sigma_s) \prod_{i \in \sigma_s} x_i \varphi_{\sigma_s}(f) \in R.$$

Remark 4.3.2 (Why anti-symmetry?). At first, the assumption that the polynomial f is anti-symmetric might seem out of place, but note that when the boundary map of a simplicial complex is defined, it is necessary to assume an order on the vertices that induces a sign on the entries of the matrix. Changing the order of the vertices is the same as changing some of the signs of the entries, and the change is equal to the sign of the permutation. In particular, for the definition of homological extension, the fact that f is anti-symmetric implies the order of the vertices has the same affect on the extension.

The following product rule for contraction will allow us to interpret homological extensions in terms of boundary maps.

Lemma 4.3.3 ([43, Lemma A.1]). *Let $R = \mathbb{K}[x_1, \ldots, x_n]$ and $S = \mathbb{K}[y_1, \ldots, y_n]$ be two polynomial rings such that R acts on S via contraction. Then for $1 \leq i \leq n$ and all $f, g \in S$ the following holds:*

$$x_i \circ fg = f(x_i \circ g) + (x_i \circ f)g.$$

In particular, if $L = x_1 + \cdots + x_n$, $f = x_1 \ldots x_n$ and $L \circ g = 0$, then

$$L \circ fg = (L \circ f)g = g \sum_{i=1}^{n} \frac{f}{x_i}.$$

We are now ready to state the first main result of this chapter.

Theorem 4.3.4 (ε-Extensions and contractions of anti-symmetric polynomials). *Let $d > 0$, Δ a d-dimensional complex such that $\tilde{H}_d(\Delta; \mathbb{K}) \neq 0$, ε a nonzero homology d-cycle,*

$L' = z_1 + \cdots + z_{d+1} \in B = \mathbb{K}[z_1, \ldots, z_{d+1}]$ and $f \in B$ a homogeneous anti-symmetric polynomial such that $L' \circ f = 0$ and the support of every monomial in f and $f \sum_{i=1}^{d+1} \frac{z_1 \ldots z_{d+1}}{z_i}$ has at most d variables. Then the ε-extension $F_\varepsilon \in R = \mathbb{K}[x_1, \ldots, x_n]$ satisfies

$$L \circ F_\varepsilon = 0,$$

where $L = x_1 + \cdots + x_n$.

Proof. First note that by the assumptions on f, for every monomial m in f we must have $|\operatorname{supp}(m)| = d$. This is the case since f is a polynomial in $d+1$ variables, and if f has a monomial m that is missing at least two variables, which we may assume without loss of generality include z_1, z_2, then the coefficient of m in $f(z_{\pi(1)}, \ldots, z_{\pi(d+1)})$ is equal to the coefficient of m in $f(z_1, \ldots, z_{d+1})$, where π is the transposition $(1\ 2)$. A contradiction to f being anti-symmetric since the sign of a transposition is -1. We conclude that for every d-face σ of Δ, the support of every monomial in $\varphi_\sigma(f)$ is a $(d-1)$–face of Δ.

Let $\varepsilon = \varepsilon(\sigma_1)\sigma_1 + \cdots + \varepsilon(\sigma_s)\sigma_s$ so that

$$F_\varepsilon = \varepsilon(\sigma_1) \prod_{i \in \sigma_1} x_i \varphi_{\sigma_1}(f) + \cdots + \varepsilon(\sigma_s) \prod_{i \in \sigma_s} x_i \varphi_{\sigma_s}(f).$$

Let $L = x_1 + \cdots + x_n \in R$ and for every j, let $x_{\sigma_j} = \prod_{i \in \sigma_j} x_i$. By Lemma 4.3.3 we know

$$L \circ F_\varepsilon = \sum_{j=1}^{s} \varepsilon(\sigma_j)(L \circ x_{\sigma_j} \varphi_{\sigma_j}(f)) = \sum_{j=1}^{s} \varepsilon(\sigma_j) \varphi_{\sigma_j}(f) \sum_{i \in \sigma_j} \frac{x_{\sigma_j}}{x_i}.$$

Now, by the assumptions on f, we know the support of every monomial in $L \circ F_\varepsilon$ is a $(d-1)$–face of Δ.

Assume now that there exists a $(d-1)$–face τ such that $m = \prod_{i \in \tau} x_i^{a_i}$ has a nonzero coefficient in $L \circ F_\varepsilon$ for some set of integers $\{a_i \colon i \in \tau\}$. We may assume without loss of generality that the d-faces of Δ that contain τ are $\sigma_1, \ldots, \sigma_k$ for some $k \le s$. We will now show that, for every $j = 1, \ldots, k$, there is exactly one monomial in $\varphi_{\sigma_j}(f)$ that contributes to the coefficient of m. Assume by contradiction that there are two monomials g_1, g_2 in $\varphi_{\sigma_j}(f)$ such that $g_1 \frac{x_{\sigma_j}}{x_{k_1}} = m = g_2 \frac{x_{\sigma_j}}{x_{k_2}}$. Since the support of m is τ, we conclude $k_1 = k_2 = \sigma_j \setminus \tau$. In particular we may cancel both fractions and get $g_1 = g_2$. This implies there is exactly one monomial in $\varphi_{\sigma_j}(f)$ that contributes to the coefficient of m, namely

$\frac{m}{x_\tau}$, where $x_\tau = \prod_{i\in\tau} x_i$. For every $j = 1, \dots k$, there is a unique monomial t_j such that $\varphi_{\sigma_j}(t_j) = \frac{m}{x_\tau}$. Denote by c_j the coefficent of t_j in f. We then have the following equation for the coefficient c_m of m in $L \circ F_\varepsilon$:

$$c_m = \sum_{j=1}^{k} \varepsilon(\sigma_j) c_j. \tag{4.2}$$

Since f is anti-symmetric, and all the t_j can be obtained by permutations of variables (the exponent vector is the same for all j), we conclude all the c_j are equal to $(-1)^{\ell_j} c$ for some fixed c and some integers ℓ_j. To determine the integer ℓ_j, note that for every j, the sign of c_j is uniquely determined by the sign of c_1, namely $(-1)^{\ell_j} = (-1)^{\ell_1} \operatorname{sgn}(\pi_j)$, where π_j is the permutation that when applied to f sends t_1 to t_j. Equation (4.2) then simplifies to

$$c_m = (-1)^{\ell_1} c \sum_{j=1}^{k} \varepsilon(\sigma_j) \operatorname{sgn}(\pi_j). \tag{4.3}$$

In particular, since $c \neq 0$ we conclude $c_m = 0$ if and only if

$$\sum_{j=1}^{k} \varepsilon(\sigma_j) \operatorname{sgn}(\pi_j) = 0. \tag{4.4}$$

To see that (4.4) holds, we will show that (4.4) is equal to (up to sign) the coefficient of τ in

$$\partial(\varepsilon(\sigma_1)\sigma_1 + \cdots + \varepsilon(\sigma_s)\sigma_s), \tag{4.5}$$

which is 0 since by definition ε is a homology cycle. For a d-chain $\lambda = \lambda(\sigma_1)\sigma_1 + \cdots + \lambda(\sigma_s)\sigma_s$, a direct computation shows the coefficient of the $(d-1)$–face τ in $\partial\lambda$ is given by

$$\sum_{j=1}^{k} \lambda(\sigma_j)(-1)^{b_j}, \tag{4.6}$$

where b_j denotes the position of the vertex of σ_j that is not in τ.

Assume $t_j = z_{j_1}^{a_1} \dots z_{j_d}^{a_d}$ for $j = 1, \dots, k$. Then the permutation π_j is also determined by the position of the vertex of σ_j that is not in τ. More specifically, let $\tau = \{i_1, \dots, i_d\}$

with $i_1 < \cdots < i_d$, for every j, the missing vertex v_j of τ in σ_j satisfies

$$i_1 < \cdots < i_{b_j - 1} < v_j < i_{b_j} < \cdots < i_d.$$

The permutation π_j can then be described as the product of the transposition $(b_1\ d+1)$ by the cycle $(b_j \cdots d+1)$. This implies $\operatorname{sgn}(\pi_j) = 1 + d + 1 - b_j$. Since $d+2$ appears for every π_j, we have shown

$$\sum_{j=1}^{k} \varepsilon(\sigma_j) \operatorname{sgn}(\pi_j) = (-1)^{d+2} \sum_{j=1}^{k} \varepsilon(\sigma_j)(-1)^{b_j},$$

so we conclude the coefficient of m is (up to sign) equal to the coefficient of τ in (4.5), which is 0 by the assumptions on ε, so the result holds. $\square$

We now list examples of anti-symmetric polynomials that satisfy the assumptions in Theorem 4.3.4. As we will see, these polynomials arise naturally in the context of Theorem 4.1.3. First, we set some notation and state a classical result of anti-symmetric polynomials.

Definition 4.3.5 (Monomial symmetric polynomials). A *partition* λ of k is a sequence of positive integers $(\lambda_1, \ldots, \lambda_s)$ such that $\lambda_1 > \cdots > \lambda_s$ and $\lambda_1 + \cdots + \lambda_s = k$. For a partition $\lambda = (\lambda_1, \ldots, \lambda_s)$ of k, the *monomial symmetric polynomial of* λ in $R = \mathbb{K}[x_1, \ldots, x_n]$ is the homogeneous polynomial

$$\mathbf{m}_\lambda = \sum_{\pi} x_{\pi(1)}^{\lambda_1} \cdots x_{\pi(s)}^{\lambda_s}, \tag{4.7}$$

where π ranges over all injective functions from $[s]$ to $[n]$.

Example 4.3.6. Let $\lambda = (4,2)$ and $n = 4$. Then injective functions from $[2]$ to $[4]$ consist of ordered choices of two distinct elements of $[4]$. In particular we have:

$$\mathbf{m}_{(4,2)} = x_1^4x_2^2 + x_1^4x_3^2 + x_1^4x_4^2 + x_2^4x_1^2 + x_2^4x_3^2 + x_2^4x_4^2 + x_3^4x_1^2 + x_3^4x_2^2 + x_3^4x_4^2 + x_4^4x_1^2 + x_4^4x_2^2 + x_4^4x_3^2.$$

Note that $\mathbf{m}_\lambda$ is a symmetric polynomial for every λ. In fact, it is not hard to show that every symmetric polynomial can be written as a linear combination of monomial symmetric polynomials (see [188, Section 7.3] for a proof in the case of symmetric monomial functions). The following is a classical fact of anti-symmetric polynomials.

Lemma 4.3.7. *Let $R = \mathbb{K}[x_1, \dots, x_n]$, where $\mathbb{K}$ is a field of characteristic 0. Then every anti-symmetric polynomial $f \in R$ can be written as $f = V([n])g$, where g is a symmetric polynomial.*

Proof. Let $f(x_1, \dots, x_n)$ be an anti-symmetric polynomial. Then $f(x_1, x_2, x_3, \dots, x_n) = -f(x_2, x_1, x_3, \dots, x_n)$ and so $f(c, c, x_3, \dots, x_n) = 0$ for all c. We then conclude $(x_1 - x_2)$ divides f. Since the same argument applies for any pair $i, j \in [n]$ we conclude $(x_i - x_j)$ divides f for every pair i, j.

Let g be the polynomial such that $f = V([n])g$. Note that $f/V([n])$ is symmetric (as a rational function), so g must be symmetric. □

We are now ready to give examples. Below, we note that the powers of the variables were not selected based on any numerical relation with the setting.

Example 4.3.8 (Monomial almost complete intersections revisited: the boundary of a simplex). The class of polynomials that will be our main source of examples comes from the failure of WLP (due to surjectivity) of monomial almost complete intersections. Here we list the first few cases. In the following examples, we assume the polynomials satisfy the assumptions in Theorem 4.3.4. This can be easily verified using, for example, Macaulay2 [73]. Moreover, we write the polynomials in the factorization from Lemma 4.3.7, in order to emphasize the polynomial g.

1. (*Coinvariant stresses* [93], $g = 1$) Let $f = V([d+1])$ be the Vandermonde determinant and
$$A = \frac{\mathbb{K}[z_1, \dots, z_{d+1}]}{(z_1 \cdots z_{d+1}, z_1^{d+1}, \dots, z_{d+1}^{d+1})}.$$
Then the failure of WLP of the algebra A shown in [148, Theorem 4.3] comes from the polynomial f being in the kernel of the map
$$\times L^\top : A_{\binom{d+1}{2}} \to A_{\binom{d+1}{2}-1}.$$

In [93] the author used the ε-extensions of $V([d])$ (without the terminology)[10] to

[10]Note that $V([d+1])$ is a homological extension of $V([d])$ by (4.1).

show that the map

$$\times L^{\top} : \Big(\frac{\mathbb{K}[x_1, \ldots, x_n]}{I_\Delta + (x_1^{d+2}, \ldots, x_n^{d+2})}\Big)_{\binom{d+2}{2}} \to \Big(\frac{\mathbb{K}[x_1, \ldots, x_n]}{I_\Delta + (x_1^{d+2}, \ldots, x_n^{d+2})}\Big)_{\binom{d+2}{2}-1}$$

is not injective for any d-dimensional complex Δ such that $\tilde{H}_d(\Delta; \mathbb{K}) \neq 0$. From the perspective of inverse systems, in [93] the author shows that for any d-dimensional $\mathbb{K}$-homology sphere with orientation ε, the ε-extension F_ε of $V([d])$ satisfies

$$(I_\Delta + (e_1, \ldots, e_{d+1}))^{-1} = (F_\varepsilon),$$

which suggests extensions of anti-symmetric polynomials might behave well with inverse systems in general, since $(e_1, \ldots, e_d)^{-1} = V([d])$. The author called the elements in (F_ε) the *coinvariant stresses* of the homology sphere Δ.

2. ($g = \mathbf{m}_{(4,2)} - \mathbf{m}_{(3,3)} + \mathbf{m}_{(2,2,2)}$) Let $f' = V([4])(\mathbf{m}_{(4,2)} - \mathbf{m}_{(3,3)} + \mathbf{m}_{(2,2,2)}) \in \mathbb{K}[x_1, x_2, x_3, x_4]$. Then f' is an anti-symmetric polynomial and f' is the polynomial that causes the algebra

$$A = \frac{\mathbb{K}[x_1, x_2, x_3, x_4]}{(x_1x_2x_3x_4, x_1^8, x_2^8, x_3^8, x_4^8)}$$

to fail the WLP due to surjectivity in degree 11.

Let $f = f'(x_1, x_2, x_3, 0)$. Then f is an anti-symmetric polynomial in $\mathbb{K}[x_1, x_2, x_3]$. If Δ is any 2-dimensional complex with nontrivial top homology, then any extension of f (with respect to a homology cycle of Δ) shows that the map

$$\times L^{\top} : \Big(\frac{\mathbb{K}[x_1, \ldots, x_n]}{I_\Delta + (x_1^8, \ldots, x_n^8)}\Big)_{12} \to \Big(\frac{\mathbb{K}[x_1, \ldots, x_n]}{I_\Delta + (x_1^8, \ldots, x_n^8)}\Big)_{11}$$

is not injective. In particular, if Δ is the boundary of a simplex with orientation ε, then a computation shows f is the ε-extension of f'.

3. ($g = \mathbf{m}_{(2,2)}$) Similar to the case above, let $f' = V([4])\mathbf{m}_{(2,2)} \in \mathbb{K}[x_1, x_2, x_3, x_4]$ and Δ the boundary of the simplex on $[5]$ with orientation ε. The ε-extension of f' is the

polynomial that causes the monomial almost complete intersection

$$A = \frac{\mathbb{K}[x_1, \dots, x_5]}{I_\Delta + (x_1^7, \dots, x_5^7)}$$

to fail the WLP due to surjectivity in degree 13. We can again take an arbitrary simplicial complex Γ with nontrivial top homology and conclude the map from degree 13 to 14 is not surjective.

Example 4.3.8 shows that, surprisingly, the failure of WLP (due to surjectivity) of the monomial almost complete intersection of the form $\frac{\mathbb{K}[x_1,\dots,x_n]}{(x_1\cdots x_n, x_1^a,\dots,x_n^a)}$ can be used to conclude maps in algebras of the form

$$\frac{\mathbb{K}[x_1, \dots, x_n]}{I_\Delta + (x_1^a, \dots, x_n^a)}$$

also fail to be surjective. The missing fact to conclude the algebra fails the WLP, is an inequality on the Hilbert series. We will come back to this later in Section 4.4.

4.3.2 Complete intersections going back to the roots: homology and differentiation (or contraction)

Suppose $f \in \mathbb{K}[x_1, \dots, x_{d+1}]$ is an anti-symmetric polynomial that causes the algebra

$$\frac{\mathbb{K}[x_1, \dots, x_{d+1}]}{(x_1 \cdots x_{d+1}, x_1^a, \dots, x_{d+1}^a)}$$

to fail the WLP due to surjectivity. In other words, $L \circ f = 0$ where $L = x_1 + \cdots + x_{d+1}$ and the highest power of a monomial in f is at most $a-1$. Assuming we know the ideal I such that $I^{-1} = (f)$ (or $I^\perp = (f)$), what can we say about the ideal I_ε such that $I_\varepsilon^{-1} = (F_\varepsilon)$ (or $I_\varepsilon^\perp = (F_\varepsilon)$), where ε is the orientation of a d-dimensional $\mathbb{K}$-homology sphere?

As an example, for a d-dimensional $\mathbb{K}$-homology sphere with orientation ε, in [93, Corollary 4.5] (see also Example 4.3.8 (1)) the author showed that $(I_\Delta + (e_1, \dots, e_{d+1}))^{-1} = (F_\varepsilon)$, where F_ε is the ε-extension of $V([d+1])$ and $e_1, \dots, e_{d+1}$ is the universal s.o.p. of Δ. We now list another example where this phenomenon appears. We will come back to this in Chapter 7.

Example 4.3.9. Let $f = V([3])(\mathbf{m}_{(4,2)} - \mathbf{m}_{(3,3)} + \mathbf{m}_{(2,2,2)}) \in \mathbb{K}[x_1, x_2, x_3]$, Δ the boundary of the simplex on 4 vertices and ε an orientation of Δ. Then the ε-extension F_ε of f factors as

$$F_\varepsilon = V([4])(\mathbf{m}_{(4,2)} - \mathbf{m}_{(3,3)} + \mathbf{m}_{(2,2,2)}) \in \mathbb{K}[x_1, x_2, x_3, x_4],$$

and the ideal I such that $I^{-1} = (F_\varepsilon)$ is $I = (e_1, e_4, h_5, h_6) \in R = \mathbb{K}[x_1, x_2, x_3, x_4]$, where h_i is the *i-th complete symmetric polynomial*, that is, the sum of every monomial of R of degree i. If we replace R by a polynomial ring in 8 variables and let Δ be the simplicial sphere[11] given by

$$\Delta = \langle 268, 168, 358, 258, 138, 357, 257, 247, 147, 137, 246, 146 \rangle,$$

then if δ is an orientation of Δ we have the δ-extension F_δ of f satisfies

$$(I_\Delta + (e_1, h_5, h_6))^{-1} = (F_\delta).$$

Note that for the boundary of the simplex, the 4-th elementary symmetric polynomial e_4 (in $\mathbb{K}[x_1, x_2, x_3, x_4]$) is also equal to its Stanley-Reisner ideal.

Example 4.3.9 shows that the elements $L = e_1, h_5, h_6 \in \mathbb{K}[x_1, \ldots, x_8]$ form a system of parameters for a certain simplicial sphere Δ. Let $A_\Delta(8) = \frac{\mathbb{K}[x_1,\ldots,x_8]}{I_\Delta + (x_1^8,\ldots,x_8^8)}$. A computation in Macaulay2 then says

$$\dim A_\Delta(8)_{11} = 504 > 498 = \dim A_\Delta(8)_{12},$$

and so e_1, h_5, h_6 is an 8-unexpected s.o.p. of Δ of total degree 12. Computational evidence suggests e_1, h_5, h_6 is an 8-unexpected s.o.p. of every 2-dimensional homology sphere, where the symmetric polynomials are taken in the corresponding rings. If true, this is a second example of a *universal* s.o.p. (see Section 3.6), at least in the context of nonacyclic complexes. Computational evidence also suggests $e_1, 3p_8 - 4h_8, h_9$ is another univeral unexpected s.o.p. for 2-dimensional complexes, where p_8 is the 8-th *power sum* symmetric polnomial, defined as $p_k = \sum_n x_i^k$.

We finish this section with the following definition which we believe encapsulates the behaviour of "universal" (unexpected) s.o.p.s.

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[11] A simple Macaulay2 computation shows Δ is a homology sphere. To see that this complex is a simplicial sphere, note that Δ is the independence complex of the graph G_2 from [10], which is known to be the boundary complex of a simplicial polytope.

Definition 4.3.10 (Simplicial complete intersections). A complete intersection ideal $I = (g_1, \ldots, g_{d+1})$ of $R = \mathbb{K}[x_1, \ldots, x_{d+1}]$ is a *simplicial complete intersection* if:

1. $x_1 + \cdots + x_{d+1}, x_1 \cdots x_{d+1} \in I$,

2. the Macaulay dual generator of R/I is the anti-symmetric polynomial F_ε which is the ε-extension of an anti-symmetric polynomial f, where ε is an orientation of the boundary of a simplex; and

3. for every d-dimensional $\mathbb{K}$-homology sphere Δ with orientation ε there exist polynomials $g'_1, \ldots, g'_{d+1}$ such that the ε-extension of f is the Macaulay dual generator of the algebra defined by the ideal $I_\Delta + (g'_1, \ldots, g'_{d+1})$.

Note that condition 3. above implies the polynomials $g'_1, \ldots, g'_{d+1}$ form an s.o.p. of I_Δ.

4.4 A simple inequality on the f-vector that causes "everything" to fail

In this section we focus on techniques to show inequalities of Hilbert functions of algebras of the form $A_\Delta(a) = \frac{\mathbb{K}[x_1,\ldots,x_n]}{I_\Delta + (x_1^a,\ldots,x_n^a)}$. This is Step 2 in the strategy mentioned in the beginning of Section 4.3. We will focus specifically on the class of algebras of the form $A_\Delta(d+2)$ where Δ is a d-dimensional complex.

Note that in an algebra of the form $A_\Delta(a)$, the nonzero monomials of degree i can be described by the possible vectors $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{N}^n$ such that the *support* $\operatorname{supp}(\mathbf{b}) = \{i \colon b_i > 0\}$ of $\mathbf{b}$ is a face of Δ, and $b_i < a$ for all i. In combinatorics, counting the number of such vectors is equivalent to counting (with restrictions) combinatorial objects called compositions, which we will now define.

A *composition* of n with l parts is a vector $(a_1, \ldots, a_l)$ of nonnegative integers such that $a_1 + \cdots + a_l = n$. Given a simplicial complex Δ and a monomial ideal $J = I_\Delta + (x_1^{a_1}, \ldots, x_n^{a_n}) \subset R$, we may view a monomial $w \notin J$ of degree i, as a composition of i with at most $\dim \Delta + 1$ nonzero parts, where exponents correspond to parts. Let $a(n, k, l)$ denote the number of compositions $\mathbf{a} = (a_1, \ldots, a_l)$ of n with l parts such that $1 \le a_i \le k$. We will need the following lemma.

Lemma 4.4.1 ([93, Lemma 6.1])**.** *Let Δ be a simplicial complex of dimension d and $I_\Delta \subset \mathbb{K}[x_1, \ldots, x_n] = R$ its Stanley-Reisner ideal. Then the Hilbert function of $A_\Delta(k+2) =$*

$\frac{R}{I_\Delta+(x_1^{k+2},\ldots,x_n^{k+2})}$ *is given by*

$$\mathrm{HF}_{A_\Delta(k+2)}(t)=\sum_{i=0}^{d} f_i a(t,k+1,i+1).$$

In particular we have

$$\mathrm{HF}_{A_\Delta(k+2)}(t-1)-\mathrm{HF}_{A_\Delta(k+2)}(t)=\sum_{i=0}^{d} f_i(a(t-1,k+1,i+1)-a(t,k+1,i+1)).$$

Proof. Nonzero monomials in $A_\Delta(k+2)$ are of the form $m=x_{i_1}^{a_1}\ldots x_{i_s}^{a_s}$ where $a_i\le k+1$, $\{i_1,\ldots,i_s\}\in\Delta$ and $s\le d+1$. In other words, for each of the f_i faces of dimension i of Δ, there are $a(t,k+1,i+1)$ nonzero monomials in $A_\Delta(k+2)$ of degree t. The result then follows by taking a sum over all faces of dimensions $0\le i\le d$. □

Let $b(n,k,l)$ be the number of compositions $\mathbf{b}=(b_1,\ldots,b_l)$ of n with l parts such that $b_i\le k$. Subtracting 1 from every part of a composition a of $n+l$ with l parts satisfying $1\le a_i\le k+1$, we get a composition b of n with l parts such that $b_i\le k$. More specifically, we have the following equality

$$a(n+l,k+1,l)=b(n,k,l)\quad\text{for every}\quad n,k,l. \tag{4.8}$$

Note that a standard stars and bars argument shows that $b(n,k,l)$ is the coefficient of x^n in the product

$$(1+x+\cdots+x^k)^l. \tag{4.9}$$

Equation (4.9) is the Hilbert series of a codimension l complete intersection generated in degree $k+1$. Thus, the Hilbert function of the monomial complete intersection

$$I=(x_1^{k+1},\ldots,x_l^{k+1})\subset\mathbb{K}[x_1,\ldots,x_l]$$

is given by

$$HF_{\frac{R}{I}}(n)=b(n,k,l). \tag{4.10}$$

Equations (4.8) and (4.10) together with the fact that artinian monomial complete intersections satisfy the SLP imply the following.

Lemma 4.4.2 ([93, Lemma 6.2]). *For any $d \geq 1$, the following holds.*

$$a(\tbinom{d+2}{2} - 1, d+1, i) \geq a(\tbinom{d+2}{2}, d+1, i) \quad \textit{for} \quad i \leq d.$$

Proof. First note that by (4.8) we have $a(t, d+1, j) = b(t-j, d, j)$ and $a(t-1, d+1, j) = b(t-j-1, d, j)$ for any $t, j > 0$.

By (4.9) the following equality holds:

$$HS_{\frac{R}{I}}(x) = b(0, d, i) + b(1, d, i)x + \cdots + b(id, d, i)x^{id},$$

where $I = (x_1^{d+1}, \ldots, x_i^{d+1}) \subset R = \mathbb{K}[x_1, \ldots, x_i]$. Now since I is a monomial complete intersection, we know the sequence of coefficients of $\mathrm{HS}_{\frac{R}{I}}(x)$ must be symmetric, and by Proposition 2.5.12, the sequence is unimodal. In particular, to prove the inequality, we only need to show that the two entries of this sequence we care about appear after its peak. Since the socle degree of R/I is id, it suffices to show that $\frac{id}{2} \leq \binom{d+2}{2} - 1 - i$. Indeed, since $d \geq i$ we have

$$\begin{aligned}\frac{(d+1)(d+2)}{2} - \frac{id}{2} - i - 1 &= \frac{(d+1)(d+2) - i(d+2)}{2} - 1\\ &= \frac{(d+2)(d+1-i)}{2} - 1\\ &\geq 0.\end{aligned}$$

We have shown that

$$a(\tbinom{d+2}{2}-1, d+1, i) = b(\tbinom{d+2}{2}-i-1, d, i) \geq b(\tbinom{d+2}{2}-i, d, i) = a(\tbinom{d+2}{2}, d+1, i) \quad \text{for} \quad i \leq d.$$

□

For the next two lemmas, we need the notion of a basic double linkage from liaison theory. Let $J \subset I \subset R = \mathbb{K}[x_1, \ldots, x_n]$ be homogeneous ideals such that $\operatorname{height} J = \operatorname{height} I - 1$. Let $\ell \in R$ be a linear form such that $J \colon \ell = J$. Then the ideal $I' := \ell I + J$ is called a *basic double link* of I. For more details on liaison theory and basic double linkage, see [111].

Lemma 4.4.3 ([111, Lemma 4.8],[148, Lemma 2.6]). *Let $J \subset I \subset R$ be homogeneous* *ideals such that* $\operatorname{height} J = \operatorname{height} I - 1$*, and a linear form* ℓ *such that* $I' = \ell I + J$ *is a basic double link of* I*. Then the following equality holds for every* j*.*

$$\mathrm{HF}_{\frac{R}{I'}}(j) = \mathrm{HF}_{\frac{R}{I}}(j-1) + \mathrm{HF}_{\frac{R}{J}}(j) - \mathrm{HF}_{\frac{R}{J}}(j-1).$$

Lemma 4.4.4 ([93, Lemma 6.4])**.** *The following equality holds for every* d

$$b\Big(\tbinom{d+2}{2}-d-1, d, d\Big) - b\Big(\tbinom{d+2}{2}-d, d, d\Big) = b\Big(\tbinom{d+2}{2}-d-1, d, d+1\Big) - b\Big(\tbinom{d+2}{2}-d-2, d, d+1\Big).$$

In particular, for

$$I_1 = (x_1^{d+1}, \ldots, x_d^{d+1}, x_{d+1}), I_2 = (x_1^{d+1}, \ldots, x_d^{d+1}, x_{d+1}^{d+1}) \subset R = \mathbb{K}[x_1, \ldots, x_{d+1}]$$

we have

$$\mathrm{HF}_{\frac{R}{I_1}}\Big(\tbinom{d+2}{2} - d - 1\Big) - \mathrm{HF}_{\frac{R}{I_1}}\Big(\tbinom{d+2}{2} - d\Big) = \mathrm{HF}_{\frac{R}{I_2}}\Big(\tbinom{d+2}{2} - d - 1\Big) - \mathrm{HF}_{\frac{R}{I_2}}\Big(\tbinom{d+2}{2} - d\Big).$$

Proof. First note that the socle degree of $\frac{R}{I_2}$ is $d(d+1)$ and since $\binom{d+2}{2} - d - 1 = \frac{d(d+1)}{2}$, by the symmetry of $\mathrm{HS}_{\frac{R}{I_2}}(t)$ we may rewrite the second statement as

$$\mathrm{HF}_{\frac{R}{I_1}}\Big(\tbinom{d+2}{2} - d - 1\Big) - \mathrm{HF}_{\frac{R}{I_1}}\Big(\tbinom{d+2}{2} - d\Big) = \mathrm{HF}_{\frac{R}{I_2}}\Big(\tbinom{d+2}{2} - d - 1\Big) - \mathrm{HF}_{\frac{R}{I_2}}\Big(\tbinom{d+2}{2} - d - 2\Big)$$

The two statements are then equivalent by (4.10).

In order to prove the lemma, let $J = (x_1^{d+1}, \ldots, x_d^{d+1}), I = J + (x_{d+1}^d) \subset R$. Since we have

1. $\operatorname{height} J = d$, $\operatorname{height} I = d + 1$
2. $J \colon x_{d+1} = J$

the ideal $I' := x_{d+1} I + J = (x_1^{d+1}, \ldots, x_{d+1}^{d+1})$ is a basic double link of I. By Lemma 4.4.3 we conclude for every j

$$\mathrm{HF}_{\frac{R}{I'}}(j) = \mathrm{HF}_{\frac{R}{I}}(j-1) + \mathrm{HF}_{\frac{R}{J}}(j) - \mathrm{HF}_{\frac{R}{J}}(j-1). \tag{4.11}$$

In particular, since x_{d+1} is a regular element of $\frac{R}{J}$ we have $\mathrm{HF}_{\frac{R}{J}}(j) - \mathrm{HF}_{\frac{R}{J}}(j-1) =$

$\mathrm{HF}_{\frac{R}{(J,x_{d+1})}}(j)$. Subtracting (4.11) applied to j and $j-1$ gives us

$$\mathrm{HF}_{\frac{R}{I'}}(j) - \mathrm{HF}_{\frac{R}{I'}}(j-1) = \mathrm{HF}_{\frac{R}{I}}(j-1) - \mathrm{HF}_{\frac{R}{I}}(j-2) + \mathrm{HF}_{\frac{R}{(J,x_{d+1})}}(j) - \mathrm{HF}_{\frac{R}{(J,x_{d+1})}}(j-1)$$

and so by (4.10) we conclude

$$b(j,d,d+1) - b(j-1,d,d+1) = \mathrm{HF}_{\frac{R}{I}}(j-1) - \mathrm{HF}_{\frac{R}{I}}(j-2) + b(j,d,d) - b(j-1,d,d). \tag{4.12}$$

Finally, the socle degree of $\frac{R}{I}$ is $d(d+1)-1$, which is an odd number and in particular

$$\mathrm{HF}_{\frac{R}{I}}\Big(\frac{d(d+1)}{2}\Big) = \mathrm{HF}_{\frac{R}{I}}\Big(\frac{d(d+1)}{2} - 1\Big).$$

The result then follows by taking $j = \binom{d+2}{2} - d$ in equation (4.12). □

We are now ready to prove the main result of this section, which should be seen as a general statement that helps us understand Step 2.

Theorem 4.4.5 ([93, Theorem 6.6])**.** *Let Δ be a d-dimensional simplicial complex where $d > 0$ and $f_{d-1} \geq f_d$, $I_\Delta \subset R = \mathbb{K}[x_1, \ldots, x_n]$ its Stanley-Reisner ideal and*

$$J = I_\Delta + (x_1^{d+2}, \ldots, x_n^{d+2}).$$

Then for $t = \binom{d+2}{2}$ we have

$$\mathrm{HF}_{\frac{R}{J}}(t-1) \geq \mathrm{HF}_{\frac{R}{J}}(t).$$

In particular, the inequality holds when Δ is a d-dimensional orientable pseudomanifold without boundary where $d > 0$.

Proof. By Lemma 4.4.1 we know the statement is equivalent to the following inequality:

$$\sum_{i=0}^{d} f_i(a(t-1,d+1,i+1) - a(t,d+1,i+1)) \geq 0. \tag{4.13}$$

By Lemma 4.4.2 we know

$$a(t-1,d+1,i+1) - a(t,d+1,i+1) \geq 0 \quad \text{for} \quad i \leq d-1,$$

hence, since $d > 0$, we may rewrite (4.13) as

$$C+f_{d-1}(a(t-1,d+1,d)-a(t,d+1,d))+f_d(a(t-1,d+1,d+1)-a(t,d+1,d+1)) \geq 0,$$

where both C and the coefficient of f_{d-1} are positive numbers. Our goal then is to show that

$$f_{d-1}(a(t-1,d+1,d)-a(t,d+1,d))+f_d(a(t-1,d+1,d+1)-a(t,d+1,d+1)) \geq 0. \tag{4.14}$$

By (4.8) we may rewrite the left side of (4.14) as

$$f_{d-1}(b(t-d-1,d,d)-b(t-d,d,d))+f_d(b(t-d-2,d,d+1)-b(t-d-1,d,d+1)). \tag{4.15}$$

By Lemma 4.4.4, equation (4.15) is equal to

$$\big(b(t-d-1,d,d)-b(t-d,d,d)\big)(f_{d-1}-f_d).$$

By Lemma 4.4.2 and since Δ satisfies $f_{d-1} \geq f_d$ we conclude

$$\mathrm{HF}_{\frac{R}{J}}(t-1) \geq \mathrm{HF}_{\frac{R}{J}}(t).$$

For the last part of the statement, a double counting argument shows that a d-dimensional pseudomanifold without boundary satisfies

$$f_{d-1} = \frac{d+1}{2} f_d,$$

this equality implies $f_{d-1} \geq f_d$ for $d > 0$. $\square$

Theorem 4.4.5 together with the results in Section 4.3 give us all the information we need regarding Step 1 and Step 2 for the class of algebras of the form $A_\Delta(d+2)$ where Δ is a d-dimensional complex such that $\tilde{H}_d(\Delta;\mathbb{K}) \neq 0$.

Theorem 4.4.6 (**Monomial almost complete intersections revisited,** [93, Theorem 6.7]**).** *Let Δ be a simplicial complex of dimension $d > 0$ such that $f_{d-1} \geq f_d$ and $\tilde{H}_d(\Delta;\mathbb{K}) \neq 0$, where $\mathbb{K}$ is a field and $J = I_\Delta + (x_1^{d+2},\ldots,x_n^{d+2}) \subset R = \mathbb{K}[x_1,\ldots,x_n]$. Then R/J fails the weak Lefschetz property.*

Proof. Let ε be a nonzero element in $\tilde{H}_d(\Delta; \mathbb{K})$ and $F_\varepsilon \in R$ the ε-extension of the Vandermonde determinant on $d+1$ variables. Then Theorem 4.3.4 implies

$$L \circ F_\varepsilon = 0,$$

where $L = x_1 + \cdots + x_n \in R$. Lemma 3.4.3 then implies the map

$$\times L^\top A_\Delta(d+2)_{\binom{d+2}{2}} \to A_\Delta(d+2)_{\binom{d+2}{2}-1}$$

is not injecive. The result then follows directly by Theorem 4.4.5. $\square$

We are now in position to tie together the probabilistic results from Section 4.2 to failure of WLP.

Theorem 4.4.7 (A scenario where failure is expected). *Let $d \geq 1$ and fix c such that $c_d < c < d+1$. Then*

$$\lim_{n\to\infty} \mathbb{P}(Y \in \mathcal{Y}_d(n, \tfrac{c}{n})\colon A_Y(d+2) \textit{ fails the WLP}) = 1.$$

Proof. Since $d > 0$, by Theorem 4.4.6 we know $A_\Delta(d+2)$ fails the WLP if Δ satisfies the following two properties:

1. $f_d \leq f_{d-1}$
2. $\tilde{H}_d(\Delta; \mathbb{K}) \neq 0$.

In particular, we conclude $\lim_{n\to\infty} \mathbb{P}(Y \in \mathcal{Y}_d(n, \frac{c}{n})\colon A_Y(d+2)$ fails the WLP$)$ is at least equal to

$$\lim_{n\to\infty} \mathbb{P}(Y \in \mathcal{Y}_d(n, \tfrac{c}{n})\colon f_{d-1} \geq f_d \text{ and } \tilde{H}_d(Y; \mathbb{K}) \neq 0). \tag{4.16}$$

The result then follows directly by Lemma 4.2.4.

$\square$

4.5 When the naive strategy pays off probabilistically

In Section 4.4 we showed that when a simplicial complex Δ is sampled from the Linial-Meshulam model $\mathcal{Y}_d(n, \frac{c}{n})$ (for n large and c in a specific threshold), it is the failure of the

WLP that should be expected for the algebra $A_\Delta(d+2)$. There are several questions that arise from this result, among them, is the question of how general are the strategy and the proof techniques used in Sections 4.3 and 4.4.

The first aspect of the broad question above is answered once we know whether the polynomials that cause failure of WLP (due to surjectivity) of monomial almost complete intersections arise from homological extensions of anti-symmetric polynomials. Although we believe this is always the case, below we give a nonexhaustive list of such polynomials.

Example 4.5.1 (Antisymmetry and failure of surjectivity). For this example, we assume the characteristic of the underlying field is 0. Given a pair (d,a) such that the monomial almost complete intersection $(x_1\cdots x_{d+2}, x_1^a,\dots,x_{d+2}^a)$ defines an algebra A that fails the WLP due to surjectivity, we will list below the polynomials such that the polynomial F causing failure of WLP of A arises as a homological extension of f (where the homology cycle is an orientation of the boundary of the simplex).

1. $(2,4)$, $f = V([3])$
2. $(2,8)$, $f = V([3])(\mathbf{m}_{(4,2)} - \mathbf{m}_{(3,3)} + \mathbf{m}_{(2,2,2)})$
3. $(2,12)$, $f = V([3])(3\mathbf{m}_{(8,4)} - 6\mathbf{m}_{(7,5)} + 8\mathbf{m}_{(6,6)} + 3\mathbf{m}_{(8,2,2)} - 3\mathbf{m}_{(7,3,2)} + 5\mathbf{m}_{(6,4,2)} - 4\mathbf{m}_{(5,5,2)} + 3\mathbf{m}_{(6,3,3)} - 2\mathbf{m}_{(5,4,3)} + 6\mathbf{m}_{(4,4,4)})$
4. $(3,5)$, $f = V([4])$
5. $(3,7)$, $f = V([4])\mathbf{m}_{(2,2)}$
6. $(3,8)$, $f = V([4])(\mathbf{m}_{(3,3)} + \mathbf{m}_{(2,2,2)})$
7. $(3,9)$, $f = V([4])(\mathbf{m}_{(4,4)} + \mathbf{m}_{(4,2,2)} + 2\mathbf{m}_{(2,2,2,2)})$

Note that all the polynomials listed above are antisymmetric, and a Macaulay2 computation shows $L\circ f = 0$ for all f listed above, where L is the sum of variables in the corresponding ring.

As a consequence for the pairs (d,a) above, let Δ be a d-dimensional complex with nontrivial top homology group. Then the multiplication map

$$\times L : \Big(\frac{\mathbb{K}[x_1,\dots,x_n]}{I_\Delta + (x_1^a,\dots,x_n^a)}\Big)_{\deg f-1} \to \Big(\frac{\mathbb{K}[x_1,\dots,x_n]}{I_\Delta + (x_1^a,\dots,x_n^a)}\Big)_{\deg f}$$

is not surjective.

Example 4.5.1 shows that Theorem 4.3.4 can be used in several settings[12], and so this handles Step 1 in the strategy mentioned earlier. For the inequality on Hilbert functions (Step 2), the goal is to translate the desired inequality on the Hilbert functions to a simple inequality on f-vectors.

Example 4.5.2 (A simple inequality for f-vectors). By Lemma 4.4.1 the Hilbert series of $A_\Delta(a)$ only depends on a, and on the f-vector of Δ. In particular, we may write the Hilbert series of $A_\Delta(a)$ as a polynomial g in the polynomial ring $\mathbb{Z}[x, f_0, \ldots, f_d]$. We now list the polynomial g for pairs (d, a) just as in Example 4.5.1. The relevant monomials for us are colored in red. Notice that the difference between coefficients (subtracting higher degree from smaller degree, including the f_i) of colored monomials of different degrees is always equal to $c(f_d - f_{d-1})$ for some positive constant c. In particular, the inequality in Step 2 holds in the settings below whenever $f_{d-1} \geq f_d$.

1. $(2, 4)$, $g = x^9 f_2 + 3\,x^8 f_2 + 6\,x^7 f_2 + x^6 f_1 + 7\,x^6 f_2 + 2\,x^5 f_1 + 6\,x^5 f_2 + 3\,x^4 f_1 + 3\,x^4 f_2 + x^3 f_0 + 2\,x^3 f_1 + x^3 f_2 + x^2 f_0 + x^2 f_1 + x\,f_0 + 1$

2. $(2, 8)$, $g = x^{21} f_2 + 3\,x^{20} f_2 + 6\,x^{19} f_2 + 10\,x^{18} f_2 + 15\,x^{17} f_2 + 21\,x^{16} f_2 + 28\,x^{15} f_2 + x^{14} f_1 + 33\,x^{14} f_2 + 2\,x^{13} f_1 + 36\,x^{13} f_2 + 3\,x^{12} f_1 + 37\,x^{12} f_2 + 4\,x^{11} f_1 + 36\,x^{11} f_2 + 5\,x^{10} f_1 + 33\,x^{10} f_2 + 6\,x^9 f_1 + 28\,x^9 f_2 + 7\,x^8 f_1 + 21\,x^8 f_2 + x^7 f_0 + 6\,x^7 f_1 + 15\,x^7 f_2 + x^6 f_0 + 5\,x^6 f_1 + 10\,x^6 f_2 + x^5 f_0 + 4\,x^5 f_1 + 6\,x^5 f_2 + x^4 f_0 + 3\,x^4 f_1 + 3\,x^4 f_2 + x^3 f_0 + 2\,x^3 f_1 + x^3 f_2 + x^2 f_0 + x^2 f_1 + x\,f_0 + 1$

3. $(2, 12)$, $g = x^{33} f_2 + 3\,x^{32} f_2 + 6\,x^{31} f_2 + 10\,x^{30} f_2 + 15\,x^{29} f_2 + 21\,x^{28} f_2 + 28\,x^{27} f_2 + 36\,x^{26} f_2 + 45\,x^{25} f_2 + 55\,x^{24} f_2 + 66\,x^{23} f_2 + x^{22} f_1 + 75\,x^{22} f_2 + 2\,x^{21} f_1 + 82\,x^{21} f_2 + 3\,x^{20} f_1 + 87\,x^{20} f_2 + 4\,x^{19} f_1 + 90\,x^{19} f_2 + 5\,x^{18} f_1 + 91\,x^{18} f_2 + 6\,x^{17} f_1 + 90\,x^{17} f_2 + 7\,x^{16} f_1 + 87\,x^{16} f_2 + 8\,x^{15} f_1 + 82\,x^{15} f_2 + 9\,x^{14} f_1 + 75\,x^{14} f_2 + 10\,x^{13} f_1 + 66\,x^{13} f_2 + 11\,x^{12} f_1 + 55\,x^{12} f_2 + x^{11} f_0 + 10\,x^{11} f_1 + 45\,x^{11} f_2 + x^{10} f_0 + 9\,x^{10} f_1 + 36\,x^{10} f_2 + x^9 f_0 + 8\,x^9 f_1 + 28\,x^9 f_2 + x^8 f_0 + 7\,x^8 f_1 + 21\,x^8 f_2 + x^7 f_0 + 6\,x^7 f_1 + 15\,x^7 f_2 + x^6 f_0 + 5\,x^6 f_1 + 10\,x^6 f_2 + x^5 f_0 + 4\,x^5 f_1 + 6\,x^5 f_2 + x^4 f_0 + 3\,x^4 f_1 + 3\,x^4 f_2 + x^3 f_0 + 2\,x^3 f_1 + x^3 f_2 + x^2 f_0 + x^2 f_1 + x\,f_0 + 1$

[12]We believe Theorem 4.3.4 always applies, that is, polynomials causing failure of the WLP for the algebra $A_\Delta(a)$, where Δ is the boundary of a simplex, are always anti-symmetric (and arising as homological extensions). We will come back to this in Chapter 7.

4. $(3,5)$, $g = x^{16} f_3 + 4\,x^{15} f_3 + 10\,x^{14} f_3 + 20\,x^{13} f_3 + x^{12} f_2 + 31\,x^{12} f_3 + 3\,x^{11} f_2 + 40\,x^{11} f_3 + 6\,x^{10} f_2 + 44\,x^{10} f_3 + 10\,x^9 f_2 + 40\,x^9 f_3 + x^8 f_1 + 12\,x^8 f_2 + 31\,x^8 f_3 + 2\,x^7 f_1 + 12\,x^7 f_2 + 20\,x^7 f_3 + 3\,x^6 f_1 + 10\,x^6 f_2 + 10\,x^6 f_3 + 4\,x^5 f_1 + 6\,x^5 f_2 + 4\,x^5 f_3 + x^4 f_0 + 3\,x^4 f_1 + 3\,x^4 f_2 + x^4 f_3 + x^3 f_0 + 2\,x^3 f_1 + x^3 f_2 + x^2 f_0 + x^2 f_1 + x\, f_0 + 1$

5. $(3,7)$, $g = x^{24} f_3 + 4\,x^{23} f_3 + 10\,x^{22} f_3 + 20\,x^{21} f_3 + 35\,x^{20} f_3 + 56\,x^{19} f_3 + x^{18} f_2 + 80\,x^{18} f_3 + 3\,x^{17} f_2 + 104\,x^{17} f_3 + 6\,x^{16} f_2 + 125\,x^{16} f_3 + 10\,x^{15} f_2 + 140\,x^{15} f_3 + 15\,x^{14} f_2 + 146\,x^{14} f_3 + 21\,x^{13} f_2 + 140\,x^{13} f_3 + x^{12} f_1 + 25\,x^{12} f_2 + 125\,x^{12} f_3 + 2\,x^{11} f_1 + 27\,x^{11} f_2 + 104\,x^{11} f_3 + 3\,x^{10} f_1 + 27\,x^{10} f_2 + 80\,x^{10} f_3 + 4\,x^9 f_1 + 25\,x^9 f_2 + 56\,x^9 f_3 + 5\,x^8 f_1 + 21\,x^8 f_2 + 35\,x^8 f_3 + 6\,x^7 f_1 + 15\,x^7 f_2 + 20\,x^7 f_3 + x^6 f_0 + 5\,x^6 f_1 + 10\,x^6 f_2 + 10\,x^6 f_3 + x^5 f_0 + 4\,x^5 f_1 + 6\,x^5 f_2 + 4\,x^5 f_3 + x^4 f_0 + 3\,x^4 f_1 + 3\,x^4 f_2 + x^4 f_3 + x^3 f_0 + 2\,x^3 f_1 + x^3 f_2 + x^2 f_0 + x^2 f_1 + x\, f_0 + 1$

6. $(3,8)$, $g = x^{28} f_3 + 4\,x^{27} f_3 + 10\,x^{26} f_3 + 20\,x^{25} f_3 + 35\,x^{24} f_3 + 56\,x^{23} f_3 + 84\,x^{22} f_3 + x^{21} f_2 + 116\,x^{21} f_3 + 3\,x^{20} f_2 + 149\,x^{20} f_3 + 6\,x^{19} f_2 + 180\,x^{19} f_3 + 10\,x^{18} f_2 + 206\,x^{18} f_3 + 15\,x^{17} f_2 + 224\,x^{17} f_3 + 21\,x^{16} f_2 + 231\,x^{16} f_3 + 28\,x^{15} f_2 + 224\,x^{15} f_3 + x^{14} f_1 + 33\,x^{14} f_2 + 206\,x^{14} f_3 + 2\,x^{13} f_1 + 36\,x^{13} f_2 + 180\,x^{13} f_3 + 3\,x^{12} f_1 + 37\,x^{12} f_2 + 149\,x^{12} f_3 + 4\,x^{11} f_1 + 36\,x^{11} f_2 + 116\,x^{11} f_3 + 5\,x^{10} f_1 + 33\,x^{10} f_2 + 84\,x^{10} f_3 + 6\,x^9 f_1 + 28\,x^9 f_2 + 56\,x^9 f_3 + 7\,x^8 f_1 + 21\,x^8 f_2 + 35\,x^8 f_3 + x^7 f_0 + 6\,x^7 f_1 + 15\,x^7 f_2 + 20\,x^7 f_3 + x^6 f_0 + 5\,x^6 f_1 + 10\,x^6 f_2 + 10\,x^6 f_3 + x^5 f_0 + 4\,x^5 f_1 + 6\,x^5 f_2 + 4\,x^5 f_3 + x^4 f_0 + 3\,x^4 f_1 + 3\,x^4 f_2 + x^4 f_3 + x^3 f_0 + 2\,x^3 f_1 + x^3 f_2 + x^2 f_0 + x^2 f_1 + x\, f_0 + 1$

7. $(3,9)$, $g = x^{32} f_3 + 4\,x^{31} f_3 + 10\,x^{30} f_3 + 20\,x^{29} f_3 + 35\,x^{28} f_3 + 56\,x^{27} f_3 + 84\,x^{26} f_3 + 120\,x^{25} f_3 + x^{24} f_2 + 161\,x^{24} f_3 + 3\,x^{23} f_2 + 204\,x^{23} f_3 + 6\,x^{22} f_2 + 246\,x^{22} f_3 + 10\,x^{21} f_2 + 284\,x^{21} f_3 + 15\,x^{20} f_2 + 315\,x^{20} f_3 + 21\,x^{19} f_2 + 336\,x^{19} f_3 + 28\,x^{18} f_2 + 344\,x^{18} f_3 + 36\,x^{17} f_2 + 336\,x^{17} f_3 + x^{16} f_1 + 42\,x^{16} f_2 + 315\,x^{16} f_3 + 2\,x^{15} f_1 + 46\,x^{15} f_2 + 284\,x^{15} f_3 + 3\,x^{14} f_1 + 48\,x^{14} f_2 + 246\,x^{14} f_3 + 4\,x^{13} f_1 + 48\,x^{13} f_2 + 204\,x^{13} f_3 + 5\,x^{12} f_1 + 46\,x^{12} f_2 + 161\,x^{12} f_3 + 6\,x^{11} f_1 + 42\,x^{11} f_2 + 120\,x^{11} f_3 + 7\,x^{10} f_1 + 36\,x^{10} f_2 + 84\,x^{10} f_3 + 8\,x^9 f_1 + 28\,x^9 f_2 + 56\,x^9 f_3 + x^8 f_0 + 7\,x^8 f_1 + 21\,x^8 f_2 + 35\,x^8 f_3 + x^7 f_0 + 6\,x^7 f_1 + 15\,x^7 f_2 + 20\,x^7 f_3 + x^6 f_0 + 5\,x^6 f_1 + 10\,x^6 f_2 + 10\,x^6 f_3 + x^5 f_0 + 4\,x^5 f_1 + 6\,x^5 f_2 + 4\,x^5 f_3 + x^4 f_0 + 3\,x^4 f_1 + 3\,x^4 f_2 + x^4 f_3 + x^3 f_0 + 2\,x^3 f_1 + x^3 f_2 + x^2 f_0 + x^2 f_1 + x\, f_0 + 1$

Examples 4.5.1 and 4.5.2 together imply that if a d-dimensional complex Δ has nontrivial top homology and $f_{d-1} \geq f_d$, the algebra $A_\Delta(a)$ fails the WLP due to surjectivity, where the pair (d, a) is one of the pairs in the examples above. Below we state the probabilistic statement, but we note that the pairs (d, a) were chosen arbitrarily, only to show that the results can be generalized. From a computational perspective, it is possible to check many more tuples (d, a).

Proposition 4.5.3 (Many more settings where failure should be expected). *Let* $(d, a) \in \{(2,4),(2,8),(2,12),(3,5),(3,7),(3,8),(3,9)\}$ *and* c *be such that* $c_d < c < d+1$. *Then*

$$\lim_{n\to\infty} \mathbb{P}(Y \in \mathcal{Y}_d(n, \tfrac{c}{n})\colon A_Y(a) \textit{ fails the WLP}) = 1$$

Proof. Example 4.5.1 shows that for a pair (d, a) in the list, and denoting by L the sum of the variables in the corresponding polynomial ring, the maps

$$\times L^\top : A_\Delta(a)_{\deg F} \to A_\Delta(a)_{\deg F - 1}$$

are not injective, where Δ is a d-dimensional complex with nontrivial top homology and F is a homological extension of the polynomial f (for the pair (d, a)) from Example 4.5.1.

Example 4.5.2 shows that for a pair (d, a) in the list, and F as above, the inequality

$$\dim A_\Delta(a)_{\deg F - 1} \geq \dim A_\Delta(a)_{\deg F}$$

holds if and only if $f_{d-1} \geq f_d$.

Both remarks above together imply the probability in the statement is bounded below by

$$\lim_{n\to\infty} \mathbb{P}(Y \in \mathcal{Y}_d(n, \tfrac{c}{n})\colon f_{d-1} \geq f_d \quad \text{and} \quad \tilde{H}_d(Y; \mathbb{K}) \neq 0).$$

Which is equal to 1 by Lemma 4.2.4. □

Remark 4.5.4 (Finding patterns from computations can be tricky). Throughout this chapter, we explored the question of whether one should expect the WLP of monomial algebras defined by an ideal of the form $I_\Delta + (x_1^a, \ldots, x_n^a)$. Our results focus mostly on what happens when the number of variables grows to infinity. We note that doing computations to find patterns in this setting can be tricky. In Figure 4.1 we see that one of the probabilities that we show converges to 1, actually decreases for values of n as large as $n = 500$. In particular, computations done in our setting with $n < 500$ could lead to the incorrect belief that the probability of failure decreases as n increases, instead of the opposite. This becomes a problem since it is not feasible to check whether an artinian monomial algebra on $n > 500$ variables satisfies the WLP.

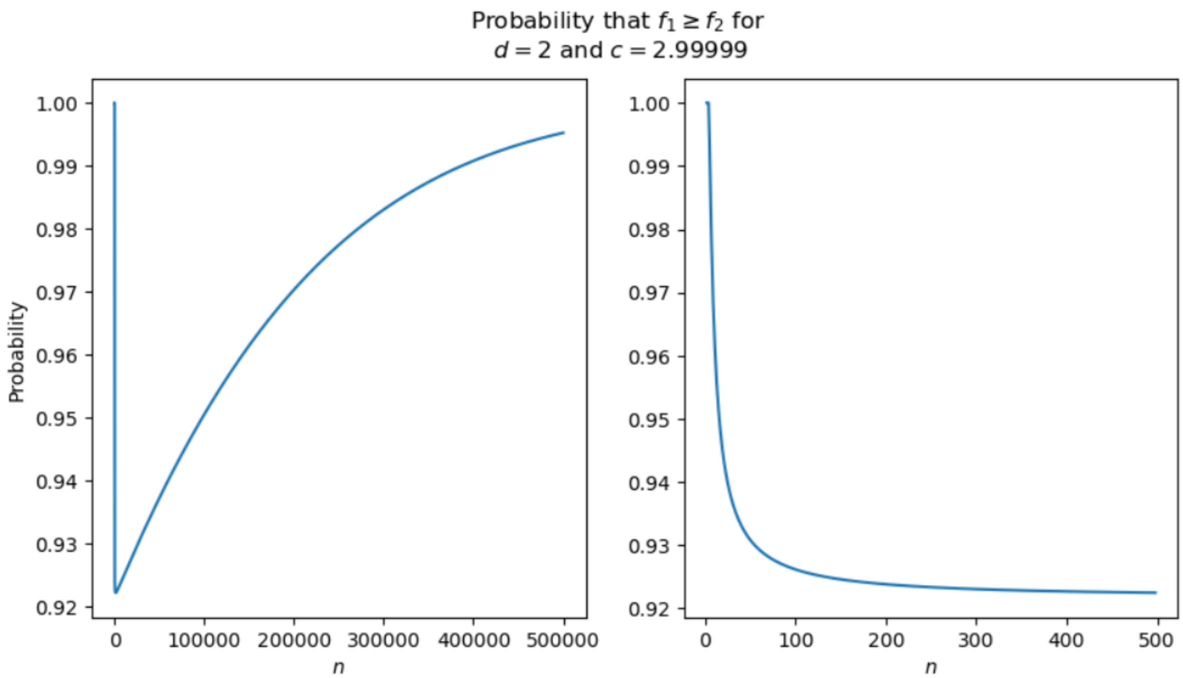

Figure 4.1: A graph of the probability (as n goes to infinity) shown in Lemma 4.2.3 to converge to 1. Notice that for values of n smaller than 500 the trend seems to be that the probability is decreasing. The full behaviour only becomes clear when we include very large values of n in the graph. This computation is easy to do considering the distribution of face numbers of random complexes is just a binomial distribution.

Chapter 5

Roller coasters: ups and downs of varying heights

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PAULO LEMINSKI – “MATERESMOFO”

So far, we have focused on studying failure of WLP of monomial algebras in order to find special systems of parameters of Cohen-Macaulay squarefree monomial ideals. The hope is that such systems of parameters would then be helpful in proving inequalities of f-vectors of classes of simplicial complexes.

In this chapter, instead of trying to find algebraic objects that could be helpful from a combinatorial perspective, we will use constructions from graph theory that shed light into the general behaviour of arbitrary artinian Gorenstein algebras. The results in this chapter are based on joint work with Lisa Nicklasson [96].

Throughout this chapter, the characteristic of every field will be assumed to be 0.

5.1 A combinatorial theory: independence polynomials of well-covered graphs

One of the many languages in which f-vectors have been studied, is in the context of independence polynomials of graphs. The main aspect of this side of f-vector theory that sets it apart, is that while f-vector theory from a more topological perspective has very strong tools for proving results about the "best case" scenario (such as triangulations of manifolds), over the years, graph theory has been shown to be a very effective area for proving results about the "worst case" scenario[1]. The most relevant example to us is the "Roller-Coaster theorem" for well-covered graphs.

Theorem 5.1.1 (**Roller Coaster theorem for well-covered graphs**, [37]). *Let $q > 0$ be an integer and π a permutation of $\{\lceil q/2\rceil, \dots, q\}$. Then there exists a well-covered graph G with independence polynomial $f_0 + f_1 t + \cdots + f_q t^q$ such that*

1. $f_0 < \cdots < f_{\lceil q/2\rceil}$ *(increasing up to the middle,* [143]*)*
2. $f_{\pi(\lceil q/2\rceil)} < f_{\pi(\lceil q/2\rceil+1)} < \cdots < f_{\pi(q)}$ *(unconstrained after the middle,* [37]*)*

The notion of "unconstrained sequences" was first introduced by Alavi et. al. [5].

Definition 5.1.2 (**Unconstrained sequences,** [5]). A collection $\mathcal{A}$ of sequences of length d is *unconstrained* if for every permutation π of d elements, there exists a sequence $(a_1, \dots, a_d)$ in $\mathcal{A}$ such that

$$a_{\pi(1)} < \cdots < a_{\pi(d)}.$$

We note that even though the original statement that the first half of independence polynomial of well-covered graphs is increasing was shown with combinatorial techniques in 2003 [143], shortly after in 2005 Hausel [81] showed the following (see [36, Lemma 2.4] for a proof of the exact statement).

Theorem 5.1.3 ([81, Theorem 6.3]). *Let $A = \mathbb{K}[x_1, \dots, x_n]/I$ be a level artinian algebra of socle degree t, where I is a monomial ideal. Let $L = x_1 + \cdots + x_n$. Then the maps $\times L^{t-2k} : A_k \to A_{t-k}$ and $\times L : A_k \to A_{k+1}$ are both injective for $k < \frac{t}{2}$.*

[1]It should be noted that there are also results from a graph theoretic perspective that show desired properties such as unimodality and log-concavity for special classes of graphs. Some of these results were mentioned in Chapters 1 and 2.

Since the Hilbert series of $A_{\mathrm{Ind}(G)}(2) = \frac{\mathbb{K}[x_1,\ldots,x_n]}{I(G)+(x_1^2,\ldots,x_n^2)}$ is the independence polynomial of G, and $A_{\mathrm{Ind}(G)}$ is level if and only if G is well-covered (or equivalently, $\mathrm{Ind}(G)$ is pure), Theorem 5.1.3 is strictly stronger than the statement in Theorem 5.1.1 that the coefficients of the independence polynomial of a well-covered graph G are increasing up to the middle.

For the second part of Theorem 5.1.1, however, the story is different. Not only there isn't an "algebraic version" of the proof, the method for proving the existence of such a graph is nonconstructive. We will now recall the notions introduced by the authors in [37], as well as some extra lemmas that we will need in order to show the main results of this chapter.

Definition 5.1.4 (**Approximate well-covered independence polynomials,** [37, Definition 1])**.** We say that a polynomial $a_q x^q + \cdots + a_1 x$ is an *approximate well-covered independence polynomial* if for every $\varepsilon > 0$, there exists a well-covered graph G with independence polynomial $1+i_1(G)x+\cdots+i_q(G)x^k$ and a real number[2] $T > 0$ such that for all $1 \leq k \leq q$,

$$\left| \frac{i_k(G)}{T} - a_k \right| < \varepsilon. \tag{5.1}$$

Given such real numbers T, ε and a graph G, we say that G is an *ε-certificate* for $a_q x^q + \cdots + a_1 x$ with *scaling factor T*.

Definition 5.1.4 highlights one interesting aspect of the results in this chapter. While the statements here are the most algebraically oriented in the thesis, we have dropped the squarefre monomial assumption for the main results. Consequently, many key arguments revolve around approximate well-covered independence polynomials and rely on esteimates and inequalities.

The following sufficient condition for a sequence to be an approximate well-covered independence polynomial appears in [37].

Theorem 5.1.5 (**A sufficient condition,** [37, Theorem 3.1])**.** *For all positive integers q and sequences of nonnegative real numbers $(a_1, \ldots, a_q)$ satisfying*

$$\frac{a_1}{\binom{q}{1}} \leq \frac{a_2}{\binom{q}{2}} \leq \cdots \leq \frac{a_q}{\binom{q}{q}}, \tag{5.2}$$

[2] In [37] the authors do not assume $T > 0$. Since the sequences we will consider all have positive coefficients, this assumption does not affect our statements.

$a_q x^q + \cdots + a_1 x$ *is an approximate well-covered independence polynomial.*

Finally, we show two lemmas that we will need in later sections. The first one is an analogue of [37, Lemma 4.1], but is about a different inequality. The reason we care about such inequalities will become clear in Section 5.2. The second lemma proves that a specific sequence is an approximate well-covered independence polynomial and is an analogue of [37, Lemma 4.2]. Our sequence however requires slightly finer estimates. The reason we need a different sequence other than the one considered in [37, Lemma 4.2] is because of the different inequalities we consider.

Lemma 5.1.6 ([96, Lemma 5.5])**.** *Let* $\sum_{i=1}^{q} a_i x^i$ *be an approximate well-covered independence polynomial with* ε*-certificate* G*. If*

$$a_k + a_{q-k+1} < a_\ell + a_{q-\ell+1}$$

then

$$i_k(G) + i_{q-k+1}(G) < i_\ell(G) + i_{q-\ell+1}(G)$$

provided that $\varepsilon > 0$ *is small enough.*

Proof. Choose $\varepsilon > 0$ such that

$$\varepsilon < \frac{1}{4}\min(|a_\ell + a_{q-\ell+1} - a_k - a_{q-k+1}|\colon 1 \leq k < \ell \leq q).$$

and let G be an ε-certificate of $a_q x^q + \cdots + a_1 x$ with scaling factor T. Then by definition we know

$$T(a_k - \varepsilon) < i_k < T(a_k + \varepsilon) \quad \text{for} \quad 1 \leq k \leq q.$$

In particular, adding two of the equations above (for k and $q - k + 1$) gives us

$$T(a_k + a_{q-k+1} - 2\varepsilon) < i_k(G) + i_{q-k+1}(G) < T(a_k + a_{q-k+1} + 2\varepsilon) \quad \text{for} \quad 1 \leq k \leq q.$$

Now we have two cases:

1. If $a_k + a_{q-k+1} - a_\ell - a_{q-\ell+1} > 0$, then we have

$$
\begin{aligned}
0 &< T(a_k + a_{q-k+1} - a_\ell - a_{q-\ell+1} - 4\varepsilon) \\
&< T(a_k + a_{q-k+1} - 2\varepsilon) - i_\ell(G) - i_{q-\ell+1}(G) \\
&< i_k(G) + i_{q-k+1}(G) - i_\ell(G) - i_{q-\ell+1}(G)
\end{aligned}
$$

where the first inequality holds by our choice of ε, and since T is positive.

2. Similarly, if $a_k + a_{q-k+1} - a_\ell - a_{q-\ell+1} < 0$, then we have

$$
\begin{aligned}
& i_k(G) + i_{q-k+1}(G) - i_{k+1}(G) - i_{q-k}(G) < \\
& T(a_k + a_{q-k+1} + 2\varepsilon) - i_\ell(G) - i_{q-\ell+1}(G) < \\
& T(a_k + a_{q-k+1} - a_\ell - a_{q-\ell+1} + 4\varepsilon) < 0
\end{aligned}
$$

□

Lemma 5.1.7 ([96, Lemma 5.6]). *Let π be a permutation of the numbers $\{\lceil \frac{q}{2} \rceil, \ldots, q\}$, for some positive integer q, and let*

$$
a_i = \begin{cases} \binom{q}{i} & \textit{for} \quad 1 \le i \le \lceil \frac{q}{2} \rceil - 1 \\ 3^q + \pi(i)\binom{q}{\lceil q/2 \rceil + 1} & \textit{for} \quad \lceil \frac{q}{2} \rceil \le i \le q. \end{cases}
$$

Then $\sum_{i=1}^{q} a_i x^i$ is an approximate well-covered polynomial, and the inequality

$$
a_k + a_{q-k+1} < a_\ell + a_{q-\ell+1}, \quad \textit{where } \lceil q/2 \rceil \le k, \ell \le q \quad \textit{and} \quad k \ne \ell
$$

holds if and only if $\pi(k) < \pi(\ell)$.

Proof. Let $c = \binom{q}{\lceil q/2 \rceil + 1}$.

We first prove that $\sum_{i=1}^{q} a_i x^i$ is an approximate well-covered polynomial. By Theorem 5.1.5 it is enough to show

$$
\frac{a_1}{\binom{q}{1}} \le \frac{a_2}{\binom{q}{2}} \le \cdots \le \frac{a_q}{\binom{q}{q}}. \tag{5.3}
$$

We have

$$\frac{a_i}{\binom{q}{i}} = \begin{cases} 1 \quad \text{for} \quad 1 \le i \le \lceil \frac{q}{2} \rceil - 1 \\ \dfrac{3^q + \pi(i)c}{\binom{q}{i}} \quad \text{for} \quad \lceil \frac{q}{2} \rceil \le i \le q. \end{cases}$$

and so we have to prove

$$\frac{3^q + \pi(k)c}{\binom{q}{k}} \le \frac{3^q + \pi(k+1)c}{\binom{q}{k+1}} \quad \text{for} \quad \lceil q/2 \rceil \le k < q$$

But in this case we have

$$\begin{aligned} \frac{3^q + \pi(k)c}{3^q + \pi(k+1)c} &\le \frac{1 + \frac{qc}{3^q}}{1 + \frac{c}{3^q}} = 1 + \frac{(q-1)c}{3^q + c} = \\ &= 1 + \frac{q-1}{1 + \frac{3^q}{c}} \le 1 + \frac{q-1}{1 + \frac{3^q}{2^q}} \le 1 + \frac{2^q(q-1)}{3^q} \le 1 + \left(\frac{2}{3}\right)^q q \end{aligned}$$

On the other hand, we also have

$$\frac{\binom{q}{k}}{\binom{q}{k+1}} = \frac{k+1}{q-k} \ge \frac{\frac{q}{2}+1}{\frac{q}{2}} = 1 + \frac{2}{q}.$$

Now for $q \ge 10$ we see that

$$\frac{3^q + \pi(k)c}{3^q + \pi(k+1)c} \le 1 + \left(\frac{2}{3}\right)^q q \le 1 + \frac{2}{q} \le \frac{\binom{q}{k}}{\binom{q}{k+1}},$$

and since we can check computationally (for $q < 10$) that

$$\frac{3^q + \pi(k)c}{3^q + \pi(k+1)c} \le \frac{1 + \frac{qc}{3^q}}{1 + \frac{c}{3^q}} \le 1 + \frac{2}{q} \le \frac{\binom{q}{k}}{\binom{q}{k+1}}$$

we conclude (5.3) holds for any q, and so the claim follows.

Next consider the inequality

$$a_k + a_{q-k+1} < a_\ell + a_{q-\ell+1}, \quad \text{for some } \lceil q/2 \rceil \le k, \ell \le q \quad \text{such that} \quad k \ne \ell.$$

By the definition of a_i this inequality states

$$3^q + \pi(k)c + \binom{q}{k-1} < 3^q + \pi(\ell)c + \binom{q}{\ell-1}$$

or equivalently

$$\binom{q}{k-1} - \binom{q}{\ell-1} < c(\pi(\ell) - \pi(k)). \tag{5.4}$$

Note that the right hand side in (5.4) is positive if and only if $\pi(k) < \pi(\ell)$. Moreover since $\lceil q/2 \rceil \leq k, \ell \leq q$, it follows that $c > \left|\binom{q}{k-1} - \binom{q}{\ell-1}\right|$. In particular, (5.4) holds precisely when $\pi(k) < \pi(\ell)$. $\square$

5.2 An algebraic insight: Nagata idealization

In this section we recall Nagata's notion of *idealization*[3] that will allow us to use the results from last section in an algebraic setting. Idealization was introduced by Nagata in 1962 [154] as a method of extending results about ideals, to the setting of modules. The construction is very natural, and as is pointed out in [7], it is unclear who first constructed an example using idealization.

Definition 5.2.1 (Nagata's idealization). Let R be a commutative ring and M an R-module. Then we denote by $R \ltimes M$ the *idealization of* M, which is the ring with underlying set $R \times M$, addition is just componentwise addition and product[4] is defined as

$$(r, m) \cdot (r', m') = (rr', rm' + r'm) \quad \text{for all} \quad r, r' \in R \quad \text{and} \quad m, m' \in M.$$

Note that R is naturally embedded in $R \ltimes M$ via the map $r \mapsto (r, 0)$.

Even though the reason Nagata first[5] considered this construction was that M is an ideal of the ring $R \ltimes M$, hence translating questions about R-modules to questions about ideals of $R \ltimes M$, the reason this construction will be useful to us is completely different. When R and (especially) M are picked carefully, the ring $R \ltimes M$ turns out to satisfy many desirable

[3]Originally, in 1962 Nagata [154] called this construction the *principle of idealization*.

[4]Note that another way of defining multiplication in $R \ltimes M$ is by defining the underlying set of $R \ltimes M$ to be the set of 2×2 matrices of the form $\begin{bmatrix} r & m \\ 0 & r \end{bmatrix}$ for all $r \in R$ and $m \in M$, and then product becomes the usual product of matrices. This is for example how Huneke [98] defines idealization. Both definitions are easily seen to be equivalent. At first sight, one could even say that the definition with matrices is more natural.

[5]In [7] Anderson and Winders mention that in the preface of [154] Nagata writes: "Among the new methods and new results given in the present book, the following four should be noted: (1) A principle, which is called the principle of idealization, and by which modules become ideals, is applied manywhere in the book."

properties (such as being Gorenstein), even though it can present very chaotic behaviour from the perspective of Hilbert series. Below we list the properties that are translated from R to $R \ltimes M$ that are relevant to us.

Proposition 5.2.2. *Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring, where $\mathbb{K}$ is a field of characteristic zero and $A = R/I$ a standard graded Cohen-Macaulay ring with canonical module ω_A. Then*

1. *If A is level of socle degree d and ω_A is generated by s elements, then $A \ltimes \omega \cong \mathbb{K}[x_1, \ldots, x_n, u_1, \ldots, u_s]/J$ is a standard graded Gorenstein $\mathbb{K}$-algebra, where $\omega = \omega_A(-d-1)$.*
2. *If A is artinian and level of socle degree d with $I^\perp = (g_1, \ldots, g_s)$, then $A \ltimes \omega \cong \mathbb{K}[x_1, \ldots, x_n, u_1, \ldots, u_s]/J$ is an artinian Gorenstein $\mathbb{K}$-algebra, where $\omega = \omega_A(-d-1)$. In this case,*

$$J^\perp = (v_1 g_1 + \cdots + v_s g_s),$$

where v_i are independent variables.

Proof. See [78, Theorem 2.77] and [131, Section 3]. □

More concretely, Proposition 5.2.2 says that questions about Gorenstein rings can, in many cases, be studied directly by considering (level) Cohen-Macaulay rings. Moreover, it gives a concrete method of generating (standard graded) Gorenstein rings from (level) Cohen-Macaulay rings. Since Cohen-Macaulay rings are, in general, not as well-behaved, these Gorenstein rings often present "bad" behaviour. The first famous example of such "bad behaviour" is the following example due to Stanley [183, Example 4.3] in 1978.

Example 5.2.3 (**h-vectors of Gorenstein rings are not necessarily unimodal).** Let $R = \mathbb{K}[x, y, z]$ and $A = R/((x, y, z)^4)$. Then A is level and

$$\mathrm{HS}_A(x) = 1 + 3x + 6x^2 + 10x^3.$$

It is clear that the socle of A is equal to A_3, hence A is level. Let $\omega = \omega_A(-4)$ be the canonical module of A (up to a shift). It can be shown that

$$\mathrm{HS}_{A\ltimes\omega}(x) = (1{+}0){+}(3{+}10)x{+}(6{+}6)x^2{+}(10{+}3)x^3{+}(0{+}1)x^4 = 1{+}13x + 12x^2 + 13x^3{+}x^4.$$

In particular, the Hilbert series of $A \ltimes \omega$ is not unimodal, even though $A \ltimes \omega$ is artinian Gorenstein by Proposition 5.2.2.

In order to fully understand the importance of Example 5.2.3, we need to consider the context in which it was stated, considering for example developments of the area around 1978. The early results of combinatorial commutative algebra relating h-vectors of complexes with h-vectors of algebras (including Stanley's g-theorem) were published between 1970 and 1980. A direct consequence of McMullen's g-conjecture (for homology spheres) is that the h-vector of a Gorenstein algebra defined by (squarefree) monomial ideals, then the h-vector is unimodal. In particular, a natural question is whether the unimodality of the h-vector of homology spheres isn't in fact a property of arbitrary Gorenstein algebras. Example 5.2.3 shows that this is not the case.

The computation done in Stanley's example for the Hilbert series is not a coincidence. The following result gives an explicit formula for the h-vector of Nagata idealizations of the form $A \ltimes \omega_A(-d-1)$, where d is the socle degree of A.

Proposition 5.2.4. *Let A be a level artinian algebra of socle degree d and canonical module ω_A. Set $\omega = \omega_A(-d-1)$. Then*

$$\mathrm{HF}_{A \ltimes \omega}(n) = \mathrm{HF}_A(n) + \mathrm{HF}_A(d-n+1).$$

Proof. See [78, Theorem 2.77] and [69, Theorem 3.2(4)]. □

Proposition 5.2.4 is the key to the main result of this chapter. The point is that for a level artinian algebra A of socle degree d, the Hilbert function of $A \ltimes \omega$ is obtained by adding the Hilbert series of A with the Hilbert series of A with coefficients in reverse, and a shift of 1. In particular, there are two settings that we should keep in mind:

1. If the Hilbert series of A is strictly increasing, then adding it to itself can end up creating a "valley" in the Hilbert series, that is, a sequence of entries in the h-vector such that
$$h_{i-1} > h_i < h_{i+1}.$$

2. If the Hilbert series of A is strictly increasing up to the middle, and in the second half it has some interesting behaviour, adding the Hilbert series to itself can result in

a sequence where the interesting behaviour is mixed with the increasing part of the sequence. This will be our strategy for the main result of this chapter.

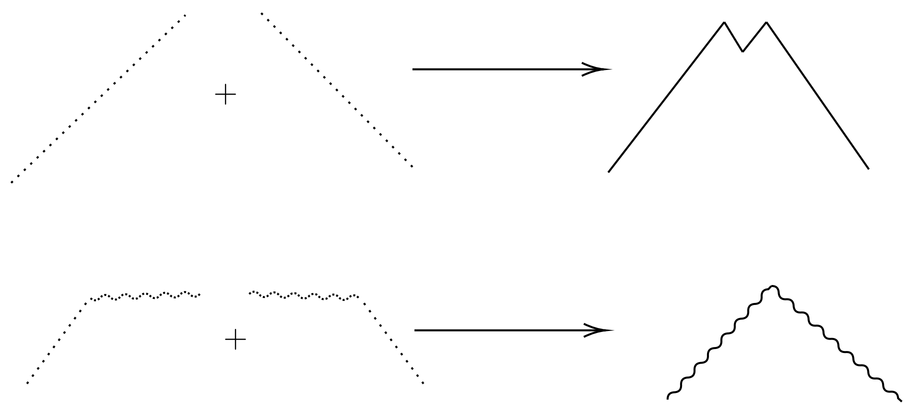


Figure 5.1: Two settings that provide us the intuition as to why Nagata idealizations are useful for generating artinian Gorenstein algebras with interesting Hilbert series. The two graphs at the top represent the scenario from Example 5.2.3, where the Hilbert series of the level artinian algebra A is strictly increasing, hence if the growth of the Hilbert series is chosen carefully, it can lead (as it does in Stanley's example) to a nonunimodal Hilbert series. The second scenario is when the Hilbert series of the level algebra A is strictly increasing in the beginning, and then it has some interesting behaviour. By carefully choosing the behaviour in the beginning and in the end, one could expect that the interesting behaviour in the end will appear in the Hilbert series of the Nagata idealization.

Considering the idea in scenario 1 very carefully, in 1995 Boij [17] showed the following.

Theorem 5.2.5 (Hilbert series of artinian Gorenstein algebras can have arbitrarily many valleys, [17, Corollary 2.9]**).** *There exists an artinian Gorenstein algebra A with socle degree $c+1$ and whose Hilbert function satisfies*

1. $\mathrm{HF}_A(2n+1) > \mathrm{HF}_A(2n)$ *for* $0 \leq 2n \leq \lfloor \frac{c-1}{2} \rfloor$
2. $\mathrm{HF}_A(2n+1) > \mathrm{HF}_A(2n+2)$ *for* $0 \leq 2n \leq \lfloor \frac{c-3}{2} \rfloor$.

In particular, the Hilbert series of an artinian Gorenstein algebra can have arbitrarily many valleys.

In this chapter we will consider scenario 2 in order to strengthen Boij's result.

Remark 5.2.6 (Examples arising from Nagata idealizations). It should be noted that the history of interesting examples arising from Nagata idealizations is longer than what we have mentioned in this section. Although we will come back to a couple of other results in this direction due to D'Alì and Venturello [38] in future sections, we refer the reader to [131, 136] for other settings where idealizations have been used in order to find interesting examples of artinian Gorenstein algebras.

5.3 A geometric motivation: Perazzo algebras

In 1851 (and in 1859[6]) Hesse [86, 87] claimed that a hypersurface $X \subset \mathbb{P}^n$ is a cone if and only if the determinant of the now called *Hessian matrix*, where the ij entry is defined as $\frac{\partial f}{\partial x_i x_j}$, has vanishing determinant[7]. One direction of this claim is true: if the hypersurface defined by f is a cone, then the determinant of the Hessian of f is zero. The other direction is only true when the projective space is of low dimension[8].

As is shown in [71], the key problem in Hesse's claim[9] is the difference between *linear independence* and *algebraic independence* of the partial derivatives of the defining polynomial f. One of the most important families of forms defining hypersurfaces that are counterexamples to Hesse's claim was introduced in 1900 by Perazzo [161] (see also [67]).

Definition 5.3.1 (**Perazzo forms).** Let $\mathbb{K}$ be a field of characteristic zero. A homogeneous polynomial $f \in \mathbb{K}[x_1, \ldots, x_n, u_1, \ldots, u_s]$ of degree d is a *Perazzo form* if there exists algebraically dependent but linearly independent polynomials $g_1, \ldots, g_s \in \mathbb{K}[x_1, \ldots, x_n]$ and $g \in \mathbb{K}[x_1, \ldots, x_n]$ such that

$$f = u_1 g_1 + \ldots u_s g_s + g.$$

At this point, we note that the connections between the Hessian matrix and the question of whether partial derivatives are algebraically independent can be understood from the remarks made in Section 2.7. More specifically, the Jacobian matrix of the partial derivatives of f is exactly the Hessian matrix of f. From Proposition 2.7.3 we see that there is a relationship between analytic spread and algebraic dependency, and from some of the results in Chapter 3 (e.g. Proposition 3.7.6) we see that analytic spread is related to Lefschetz

[6]Usually it is only mentioned that the claim was made in 1851, but in [68] the authors mention the second time the claim was made, in [87]

[7]Since the determinant is a polynomial, this means the determinant, as a polynomial, is identically zero.

[8]We refer the reader to [122] for more details and history on Hesse's claim.

[9]In [122] the author mentions that the first person to notice a mistake in Hesse's proofs was Weistrass.

properties.

In the setting of artinian Gorenstein algebras there is a more direct relationship between Lefschetz properties and Hessian matrices. A theorem of Watanabe [129, 200] relates the SLP of artinian Gorenstein algebras with "higher" Hessians of homogeneous polynomails. Although we will not need the full strength of this result, below we state a consequence that is of relevance to us.

Proposition 5.3.2. *Let $f \in R$ be a Perazzo form of degree d and $A = R/I$ the artinian Gorenstein algebra such that $I^\perp = (f)$. Then A fails the SLP due to the multiplication map from degree 1 to d failing to have full rank.*

Algebras defined by an ideal such that the inverse system is generated by a Perazzo polynomial are called *Perazzo algebras*. In view of Proposition 5.3.2, Perazzo algebras are an interesting family of algebras, as not only do they fail the SLP (a property which, as was mentioned in previous chapters, algebras are often expected to satisfy), but the failure happens in the multiplication map by the largest possible (nontrivial) power of a linear form.

One important type of question that has been recently studied from different perspectives is to understand properties of the Hilbert series of Perazzo algebras, as well as to understand which Perazzo algebras satisfy the *weak* Lefschetz property. The main results of this chapter show that combinatorial tools are extremely useful in order to explore these problems.

5.4 Simplicial Perazzo forms and Koszulity: graphs and flag complexes

We now start tying the notions from previous sections together in order to prove the main results of this chapter.

5.4.1 Simplicial Perazzo forms

Recall that given a simplicial complex Δ we define the algebra $A_\Delta(2) = \frac{\mathbb{K}[x_1,\ldots,x_n]}{I_\Delta + (x_1^2,\ldots,x_n^2)}$. Note that $\mathrm{HF}_{A_\Delta(2)}(i) = f_{i-1}$, and if we assume that Δ is pure, then $A_\Delta(2)$ is a level algebra with socle generated by $\{\prod_{i\in\sigma} x_i \colon \sigma \text{ is a facet of } \Delta\}$. For our context in this chapter, this means for every pure simplicial complex Δ of dimension $d-1$ we may define an artinian Gorenstein algebra of socle degree $d+1$ by taking the Nagata idealization of $A_\Delta(2)$ and its canonical module (up to shift) $\omega = \omega_{A_\Delta(2)}(-d-1)$ (note that if Δ is $(d-1)$–dimensional, the socle degree of $A_\Delta(2)$ is d). To the best of our knowledge, the first time this algebra was considered, was in 2018 by Gondim and Zappalà [69], and then later in 2023 by D'Alì and Venturello [38].

Definition 5.4.1 (Simplicial (Perazzo) forms). Let $\Delta = \langle \sigma_1, \ldots, \sigma_s \rangle$ be a pure simplicial complex of dimension $d-1$ and set $x_{\sigma_i} = \prod_{j \in \sigma_i} x_j \in R = \mathbb{K}[x_1, \ldots, x_n]$. The *simplicial form* of Δ is the polynomial

$$F_\Delta = u_1 x_{\sigma_1} + \cdots + u_s x_{\sigma_s} \in \mathbb{K}[x_1, \ldots, x_n, u_1, \ldots, u_s].$$

If the monomials $x_{\sigma_1}, \ldots, x_{\sigma_s}$ are algebraically dependent, we call F_Δ the *simplicial Perazzo form* of Δ.

Note that since the socle of $A_\Delta(2)$ is generated by $x_{\sigma_1}, \ldots, x_{\sigma_s}$, by Proposition 5.2.2 F_Δ is the Macaulay dual generator of $A_\Delta(2) \ltimes \omega$, where $\omega = \omega_{A_\Delta(2)}(-d-1)$. In particular, we have the following result.

Lemma 5.4.2 (Simplicial Perazzo forms). *Let $\Delta = \langle F_1, \ldots, F_s \rangle$ be a pure d-dimensional simplicial complex such that the multiplication map $\times L^d : A(\Delta)_1 \to A(\Delta)_{d+1}$ is not surjective. Then $F(\Delta)$ is a Perazzo form, which we call the* simplicial Perazzo form *of Δ.*

Proof. By [92, Theorem 29] the multiplication map $\times L : A(\Delta)_1 \to A(\Delta)_{d+1}$ is exactly the log-matrix of $u_{F_1}, \ldots, u_{F_s}$. So if the map is not surjective, then by [192, Lemma 4.2] the monomials $u_{F_1}, \ldots, u_{F_s}$ are algebraically dependent, which makes form $F(\Delta)$ Perazzo. $\square$

5.4.2 Koszul algebras

We now define a class of algebras that is often expected to have good behaviour, and is implicitly related to several questions in graph theory.

Definition 5.4.3. Let $R = \mathbb{K}[x_1, \ldots, x_n]$ and a homogeneous ideal $I \subset R$. The $\mathbb{K}$-algebra $A = R/I$ is called a *Koszul algebra* if the field $\mathbb{K}$ has a linear (possibly infinite) free resolution over the ring A.

Koszul algebras were introduced by Priddy in 1970 [165]. One of the most fundamental properties of Koszul algebras is the following (see [159, Exercise 34.2]).

Proposition 5.4.4. *Let $R = \mathbb{K}[x_1, \ldots, x_n]$, $I \subset R$ a homogeneous ideal and $A = R/I$ a Koszul algebra. Then I is generated in degrees 1 and 2.*

In particular, if $I \subset R$ is a squarefree monomial ideal, then $I = I(G)$ for some graph G.

The converse of Proposition 5.4.4 is also true: $R/I(G)$ is Koszul for every graph G.

Since their introduction, Koszul algebras have always been an interesting topic of research due to the several nice properties that they possess. As an example, in 2001, Conca, Trung and Valla [31, Theorem 3.1] showed that a polynomial $p(x) = ax^2 + bx + 1$ is the Hilbert series of a Koszul algebra if and only if $p(x)$ has real roots. A (trivial) consequence of this property, that follows from a well-known result of Newton, is that the Hilbert series of an artinian Koszul algebra of socle degree 2 is unimodal. Although this unimodality is trivial, the real rootedness (which is strictly stronger) suggests the Hilbert series of Koszul algebras are nicer objects than arbitrary Hilbert series. This is indeed true, at least to some extent, since there are concrete inequalities that the Hilbert series of Koszul algebras must satisfy [164, Chapter 7].

This line of thought has led several conjectures about Koszul algebras to be made throughout the years, including Question 1.1.10. Other examples include conjectures made by Migliore and Nagel in 2013 [145, Conjectures 4.1, 4.5], on the Lefschetz properties of artinian Gorenstein algebras generated by quadrics. Counterexamples to these conjectures were first presented by Gondim and Zappalà in 2018 [69].

The common theme between the counterexamples found by Gondim and Zappalà [69], as well as the counterexamples found by D'Alì and Venturello [38], is that they both arise from flag complexes, or equivalently, from independence complex of graphs. More concretely, both sets of authors considered the Nagata idealization of algebras of the form $A_{\mathrm{Ind}(G)}(2)$, where $\mathrm{Ind}(G)$ is a pure simplicial complex. In particular, the following is one of the main results of [38].

Theorem 5.4.5 ([38, Corollary 8.2][10]). *Let $\Delta = \langle \sigma_1, \ldots, \sigma_s \rangle$ be a pure flag d-dimensional complex and $A = R/I$ the algebra such that $I^{\perp} = (F_\Delta)$, where*

$$F_\Delta = u_1 \prod_{i \in \sigma_1} x_i + \cdots + u_s \prod_{i \in \sigma_s} x_i \in \mathbb{K}[x_1, \ldots, x_n, u_1, \ldots, u_s].$$

[10] A similar result, with a weaker assumption is also claimed in [69]. In [38] the authors mention a counterexample to the stronger statement, and clarify that this mistake does not affect the main results of [69].

Then Δ is Cohen-Macaulay over the field $\mathbb{K}$ of characteristic zero if and only if A is an artinian Gorenstein Koszul algebra.

The perspective taken in [38] in order to prove Theorem 5.4.5 is very topological. In order to prove the main results of this chapter, we will take a more graph theoretic perspective.

5.5 Graph theory as a lower bound to bad behaviour in artinian Gorenstein algebras

We are now ready to put the pieces from previous sections together.

Theorem 5.5.1 (**A Roller Coaster theorem for (Perazzo) Gorenstein algebras,** [96, Theorem 5.7]**).** *For every $d \geq 2$, the collection of all sequences $(h_1, \dots, h_{\lfloor d/2 \rfloor})$ such that $(h_0, h_1, \dots, h_d)$ is the h-vector of a (flag) simplicial Perazzo algebra is unconstrained.*

Proof. Let π be a permutation of $\{\lceil d/2 \rceil, \dots, d-1\}$. Note that we may also view π as a permutation of $\{1, \dots, \lfloor d/2 \rfloor\}$. Let $p = \sum_{i=1}^{d-1} a_i x^i$ be the approximate well-covered polynomial defined in Lemma 5.1.7, where we take $q = d-1$. Let G be an ε-certificate for p with scaling factor T, where we have chosen $\varepsilon > 0$ so that $2\varepsilon < a_{d-1} - a_1$, and small enough so that we can apply Lemma 5.1.6. Note that the maximal independent sets of G have size $d-1$, and so $\Delta = \mathrm{Ind}(G)$ is $(d-2)$–dimensional and $A(\Delta)$ is a level algebra of socle degree $d-1$. The assumptions on ε and the definition of approximate well-covered independence polynomials imply

$$i_1(G) < T(a_1 + \varepsilon) < T(a_q - \varepsilon) < i_{d-1}(G).$$

In particular, the independence complex of G has more facets than vertices. Since the facets $F_1, \dots, F_{i_{d-1}(G)}$ of the independence complex of G correspond to socle elements of $A(\Delta)$, and hence generators $u_{F_1}, \dots, u_{F_{i_{d-1}(G)}}$ of the inverse system of $I(G) + (x_1^2, \dots, x_{i_1(G)}^2)$, we conclude the polynomial

$$F(G) = \sum_{j=1}^{i_{d-1}(G)} x_j u_{F_j}$$

is a flag simplicial Perazzo form, as the u_{F_j} are algebraically dependent by Lemma 5.4.2 and a straightforward cardinality argument. Let A be the algebra such that $F(G)$ is the

Macaulay dual generator of A. By Proposition 5.2.4, and since $\dim_{\mathbb{K}} A(\Delta)_j = i_j(G)$, we conclude the Hilbert function of A is given by:

$$h_k = i_k(G) + i_{d-k}(G),$$

where $i_d(G) = 0$ since the maximal independent subsets of G have size $d-1$.

For k, ℓ distinct such that $\lceil (d-1)/2 \rceil \leq k, \ell \leq d-1$, by Lemma 5.1.7 the sequence $a_1 + a_{d-1}, \ldots, a_{\lfloor d/2 \rfloor} + a_{d-\lfloor d/2 \rfloor}$ satisfies the inequality

$$a_k + a_{d-k} < a_\ell + a_{d-\ell} \quad \text{if and only if} \quad \pi(k) < \pi(\ell).$$

By Lemma 5.1.6 these inequalities are transfered to our Hilbert function h_k. By symmetry we may take analogous inequalities on the first half of the sequence $h_0, h_1, \ldots, h_d$ instead.

Given a permutation π, we have shown that there exists an Artinian Gorenstein algebra A such that the Hilbert function of A satisfies the inequalities prescribed from π. Since the A we constructed is a flag simplicial Perazzo algebra, the result follows. □

Theorem 5.5.1 is an example (of many) of how combinatorics can be used in order to generate interesting examples in commutative algebra. Nagata idealization allows us to generally translate results about Lefschetz properties of level monomial algebras, to find examples of Gorenstein algebras with interesting behaviour with respect to Lefschetz properties. This can be seen in the following observation, originally due to Dao and Nair [40].

Proposition 5.5.2 ([40, Proposition 5.1])**.** *Let $\mathbb{K}$ be a field of characteristic zero and A a standard graded artinian level $\mathbb{K}$-algebra of socle degree d with canonical module ω. Set $\tilde{A} = A \ltimes \omega(-d-1)$. Assume further that* $\dim A_i + \dim A_{d-i} \leq \dim A_{i-1} + \dim A_{d-i+1}$. *If the multiplication map $\times L : A_{i-1} \to A_i$ is not surjective for any linear form L, then $\tilde{A}$ fails the WLP due to surjectivity in degree $i-1$.*

Concretely, if A is a level monomial algebra that fails WLP due to surjectivity in a given degree, then the Nagata idealization $\tilde{A}$ as above must also fail the WLP due to surjectivity by Proposition 5.5.2.

Remark 5.5.3 (Other results in [96]**).** In order to simplify notation, given a graph G let $A(G,d)$ denote the algebra

$$\frac{\mathbb{K}[x_1, \ldots, x_n]}{I(G) + (x_1^d, \ldots, x_n^d)},$$

where $\mathbb{K}$ is a field of characteristic zero. In [96], besides Theorem 5.5.1, the authors also study the WLP of $A(G, d)$ for various values of d when G is a whiskered graph. In particular, it is shown that if G is the graph obtained by whiskering a graph H, and H has a high independence number (relative to number of vertices), then $A(G, 2)$ fails the WLP [96, Corollary 3.12]. This result, and its (conjectural) implications, should be considered together with the results in Chapter 4, where algebras of the form $A(G, 2)$ give another setting where failure is common. We will come back to the conjectural implications of the results in [96] in Chapter 7.

Chapter 6

Spherical complexes

"Foi muito mais do que eu queria.
Foi bem mais forte.
Durou muito mais que um dia.
Mesmo assim,
quem ia imaginar?"

TIÊ – "MEXEU COMIGO"

In Chapters 3 and 4 we studied the failure of Lefschetz properties of artinian algebras defined by monomial ideals, which, as mentioned in Section 2.5 (in the nonmonomial case[1]), can be very effective at providing information on the h-vector of simplicial complexes. In this chapter, we will focus on understanding the free resolution of a special class of complexes, which we will introduce.

One of the motivations for understanding free resolutions of squarefree monomial ideals is that a minimal free resolution is a finer invariant than the h-vector, that is, knowing the graded Betti numbers of R/I_Δ is enough to compute the h-vector of Δ (see e.g. Example 2.6.6), while the converse is not true.

The results in this chapter are based on joint work with Sara Faridi [57].

6.1 A surprising topological property of trees

In 2006, Ehrenborg and Hetyei [46] developed tools to characterize the homotopy type of several classes of complexes in combinatorial topology. As an application, they showed the following.

Theorem 6.1.1 ([46, Corollary 6.1])**.** *Let G be a forest, that is, a graph without cycles. Then $\operatorname{Ind}(G)$ is either contractible or homotopy equivalent to a sphere.*

[1] We will come back to the role of the monomial setting in understanding h-vectors in Chapter 7.

Shortly after in 2007, Marietti and Testa [130] described the dimension of the sphere $\mathrm{Ind}(G)$ is homotopy equivalent to (when it is not contractible), in terms of graph theoretical invariants of the forest G. Since the alternating sum of Betti numbers of $\mathrm{Ind}(G)$ (Euler characteristic) is equal to the alternating sum of the f-vector of $\mathrm{Ind}(G)$, Theorem 6.1.1 implies an interesting inequality on the difference between the number of independent sets of a forest of even and odd size.

Another interesting consequence of Theorem 6.1.1 is when G is a forest, *every* induced subcomplex of $\mathrm{Ind}(G)$ is either contractible or homotopy equivalent to a sphere. This is simply because every induced subcomplex of $\mathrm{Ind}(G)$ is equal to $\mathrm{Ind}(H)$ for some induced subgraph H of G (which must also be a forest).

The topological property mentioned above is an extremely strong topological restriction. The reader should compare this topological property with the notion of homology sphere (Definition 2.2.4): while for a homology sphere it is required that only the *top* homology group is nonzero, and it has dimension 1, the topological property satisfied by independence complexes of forests only asks that *at most one* homology group is nonzero, dropping the requirement on the homological index of the nonzero group. The other part of the definition of a homology sphere describes the homology of a certain class of subcomplexes (links). The independence complex of a forest on the other hand gives the possible homotopy types of *every induced subcomplex*. As the next (well-known) proposition shows, this class of subcomplexes includes all links (for flag complexes). We recall that given a graph G and a vertex v of G, the *(closed) neighborhood* of v is the set $N[v] = \{u \colon \{u, v\} \in G\}$ of vertices of G adjacent to v.

Proposition 6.1.2. *Let G be a graph and v a vertex of G. Then $\mathrm{link}_{\mathrm{Ind}(G)}(v) = \mathrm{Ind}(G \setminus N[v])$. In particular, in a flag complex, links are induced subcomplexes.*

In summary, the topological property imposed by Theorem 6.1.1 on the independence complex of a forest is neither weaker nor stronger than the notion of homology spheres:

1. The definition of homology sphere imposes a stronger restriction on homology groups, since the top homology group is the only group allowed to be nonzero,

2. The definition of homology sphere imposes conditions only on links, while the property from Theorem 6.1.1 is for every induced subcomplex.

From a completely different perspective, in 2002 Faridi [53] introduced a generalization of the notion of trees and forests from graph theory, to simplicial complexes of arbitrary dimension[2].

Definition 6.1.3 (**Leaves, simplicial trees and forests,** [53, Section 3]**).** Let $\Delta = \langle \sigma_1, \ldots, \sigma_s \rangle$ be a simplicial complex. A facet σ_i of Δ is called a *leaf* if either σ_i is the only facet of Δ, or there exists another facet σ_j of Δ such that $\sigma_i \neq \sigma_j$ and

$$\sigma \cap \sigma_i \subseteq \sigma_i \cap \sigma_j \quad \text{for all} \quad \text{facets } \sigma \neq \sigma_i \text{ of } \Delta.$$

If every nonempty subcomplex of Δ has a leaf (including Δ itself), we say Δ is a *(simplicial) forest*. If moreover Δ is connected, we say Δ is a *(simplicial) tree*.

Simplicial forests satisfy a similar property to Proposition 6.1.2.

Proposition 6.1.4 ([53, Lemma 1], [56, Theorem 2.5], [55, Lemma 7])**.** *Let Δ be a simplicial forest and F a face of Δ. Then $\operatorname{link}_\Delta(F)$ and $\operatorname{del}_\Delta(F)$ are simplicial forests. In particular, every induced subcomplex of Δ is a simplicial forest.*

Finally in 2014, Erey and Faridi [52] showed that independence complexes of simplicial forests satisfy a similar topological restriction as in the graph setting. More specifically, they showed the following.

Theorem 6.1.5 ([52, Theorem 3.5])**.** *Let Γ be a simplicial forest. Then the multigraded Betti numbers of $I(\Gamma)$ are either 0 or 1. Moreover, if for some $\mathbf{a} = (a_1, \ldots, a_n)$ we have $\beta_{i,\mathbf{a}}(R/I(\Gamma)) \neq 0$, then $\beta_{j,\mathbf{a}}(R/I(\Gamma)) = 0$ for every $j \neq i$.*

We note that the perspective the authors take in [52] is of computing multigraded Betti numbers, hence the language used in the statement. Hochster's formula directly shows that for a simplicial forest Γ, there exists at most one integer $i \geq 0$ such that $\tilde{H}_i(\operatorname{Ind}(\Gamma); \mathbb{K}) \neq 0$. Moreover, when such an i exists, we have $\tilde{H}_i(\operatorname{Ind}(\Gamma); \mathbb{K}) = \mathbb{K}$.

Proposition 6.1.4 then says that, at the level of homology groups, the independence complex of a simplicial forest satisfies the same topological restrictions as the independence complex of a forest (from Theorem 6.1.1). Since the invariants we care about in this

[2] We note that there is not a unique notion that generalizes the notion of a tree/forest from graph theory. We will, however, only use the one introduced by Faridi.

chapter do not see the (subtle) difference between homotopy type, homotopy groups and homology groups, this distinction will not play a role, and we will treat both results as the same[3].

6.2 Ternary graphs and spherical complexes

In 2022 Kim [109] extended Theorem 6.1.1 to the largest possible class of graphs, solving an earlier conjecture by Engström, which was originally motivated by previously asked questions of Kalai and Meshulam. A graph is said to be *ternary* if it does not have induced cycles of length divisible by 3.

Theorem 6.2.1 ([109, Theorem 1.1])**.** *A graph G is ternary if and only if every induced subcomplex of* $\mathrm{Ind}(G)$ *is either contractible or homotopy equivalent to a sphere.*

Example 6.2.2 (Why 3?). Before moving on, we note that the reason 3 plays a key role in Theorem 6.2.1 can be explained by looking at paths and cycles. From Theorem 6.1.1 we know that the independence complex of every path is either contractible, or homotopy equivalent to a sphere. Recall that by Theorem 2.1.7 we have a long exact sequence

$$\cdots \to \tilde{H}_i(\mathrm{link}_\Delta(v);\mathbb{K}) \to \tilde{H}_i(\mathrm{del}_\Delta(v);\mathbb{K}) \to \tilde{H}_i(\Delta;\mathbb{K}) \to \tilde{H}_{i-1}(\mathrm{link}_\Delta(v);\mathbb{K}) \to \cdots$$

where Δ is the independence complex $\mathrm{Ind}(C_n)$ of the cycle graph on n vertices, and v is a vertex of Δ. Since by Proposition 6.1.2 taking links corresponds to deleting closed neighborhoods, we conclude that $\mathrm{link}_\Delta(v) = \mathrm{Ind}(P_{n-3})$ and $\mathrm{del}_\Delta(v) = \mathrm{Ind}(P_{n-1})$, where P_n is the path graph on n vertices.

Now in order to determine the homology of $\mathrm{Ind}(P_n)$, let v be a vertex adjacent to a leaf of P_n, and note that $\mathrm{link}_{\mathrm{Ind}(P_n)}(v) = \mathrm{Ind}(P_{n-3})$ and $\mathrm{del}_{\mathrm{Ind}(P_n)}(v) = \mathrm{Ind}(P_n - v)$. Since the leaf u adjacent to v is an isolated vertex of $P_n - v$, we conclude $\mathrm{Ind}(P_n - v)$ is a cone (with apex u), and so must be acyclic. The Mayere-Vietoris sequence from Theorem 2.1.7 then says

$$\tilde{H}_i(\mathrm{Ind}(P_n);\mathbb{K}) \cong \tilde{H}_{i-1}(\mathrm{Ind}(P_{n-3});\mathbb{K}) \quad \text{for all} \quad i.$$

Since we may repeat this process as long as $n > 3$, we see that in order to understand the homology of $\mathrm{Ind}(P_n)$ it is enough to understand the homology of $\mathrm{Ind}(P_a)$ where $a \equiv n$

[3]It should be noted, however, that almost certainly there is a homotopy type statement for simplicial forests similar to Theorem 6.1.1.

mod 3. This explains why 3 plays an important role in Theorem 6.2.1. To see why the case $0 \equiv n$ mod 3 is different, notice that a more careful analysis shows that (see for example Kozlov's original paper [113, Proposition 4.6])

$$\mathrm{Ind}(P_n) \cong \begin{cases} S^{k-1} & \text{if} \quad n = 3k \\ \text{a point} & \text{if} \quad n = 3k+1 \\ S^k & \text{if} \quad n = 3k+2. \end{cases}$$

In particular, for $n = 3k$ the Mayer-Vietoris sequence applied to $\mathrm{Ind}(C_n)$ gives us

$$\cdots \to \tilde{H}_i(S^{k-2};\mathbb{K}) \to \tilde{H}_i(S^{k-1};\mathbb{K}) \to \tilde{H}_i(\mathrm{Ind}(C_n);\mathbb{K}) \to \tilde{H}_{i-1}(S^{k-2};\mathbb{K}) \to \cdots$$

and so for $i = k-1$ we have

$$0 \to \mathbb{K} \to \tilde{H}_{k-1}(\mathrm{Ind}(C_n);\mathbb{K}) \to \mathbb{K} \to 0,$$

so $\tilde{H}_{k-1}(\mathrm{Ind}(C_{3k};\mathbb{K}) \cong \mathbb{K} \oplus \mathbb{K}$ and in particular we conclude $\mathrm{Ind}(C_{3k})$ does not have the homology of a sphere.

At this point, we notice that the class of complexes Δ such that

1. Δ has the homology of a sphere, and

2. every induced subcomplex is either acyclic or has the homology of a sphere

contains some interesting combinatorially defined complexes. Moreover, as mentioned in the previous section, the similarity to the definition of homology spheres suggests such complexes might behave well with respect to standard operations and invariants that are well studied.

With this in mind, in [57] the authors introduce the class of *spherical complexes*.

Definition 6.2.3 (**Spherical complexes,** [57, Definition 2.3])**.** Let $\mathbb{K}$ be a field. A simplicial complex Δ is $\mathbb{K}$-*spherical* if

1. $\tilde{H}_i(\Delta;\mathbb{K}) = \begin{cases} 1 & \text{for at most one value of } i \\ 0 & \text{otherwise;} \end{cases}$

2. for every $v \in V$ the two complexes $\mathrm{link}_\Delta(v)$ and $\mathrm{del}_\Delta(v)$ are $\mathbb{K}$-spherical.

We note that since every deletion is $\mathbb{K}$-spherical, we conclude that every induced subcomplex is also $\mathbb{K}$-spherical. The definition includes the fact that every link is also $\mathbb{K}$-spherical as this is true in every example we know, and it simplifies several proofs.

Furthermore, although the definition of $\mathbb{K}$-spherical depends on the underlying field, every known example of a spherical is spherical over *every* field $\mathbb{K}$. In other words, it is possible that a complex Δ is $\mathbb{K}$-spherical over some field $\mathbb{K}$ if and only if it is $\mathbb{K}$-spherical over every field $\mathbb{K}$.

Theorems 6.1.5 and 6.2.1 together with Propositions 6.1.2 and 6.1.4 show that independence complexes of ternary graphs and simplicial forests are examples of spherical complexes.

6.3 Filtrations of spherical complexes: visualizing Mayer-Vietoris sequences

The definition of spherical complexes requires that every subcomplex obtained by repeatedly applying links and deletion operations must still be spherical. This recursive aspect of the definition allows us to apply Mayer-Vietoris techniques very effectively (similarly to the computation done in Example 6.2.2 for independence complex of cycles and paths). In this section we introduce the required concepts in order to organize the data arising from several Mayer-Vietoris sequences together.

The key idea is that we may view a Mayer-Vietoris sequence as a branching point in a binary tree. Before formalizing this idea, we need the following lemma.

Lemma 6.3.1. *Let Δ be a $\mathbb{K}$-spherical complex and v a vertex. Then at least one of the following holds*

(1) $\tilde{H}_j(\Delta;\mathbb{K}) \cong \tilde{H}_j(\mathrm{del}_\Delta(v);\mathbb{K})$ *and* $\tilde{H}_j(\mathrm{link}_\Delta(v);\mathbb{K}) = 0$ *for every j,*

(2) $\tilde{H}_j(\Delta;\mathbb{K}) \cong \tilde{H}_{j-1}(\mathrm{link}_\Delta(v);\mathbb{K})$ *and* $\tilde{H}_j(\mathrm{del}_\Delta(v);\mathbb{K}) = 0$ *for every j or*

(3) $\tilde{H}_j(\mathrm{link}_\Delta(v);\mathbb{K}) \cong \tilde{H}_j(\mathrm{del}_\Delta(v);\mathbb{K})$ *and* $\tilde{H}_j(\Delta;\mathbb{K}) = 0$ *for every j*

Proof. By the definition of spherical complex, we have two possible cases. If Δ is acyclic, then Theorem 2.1.7 implies

$$\tilde{H}_j(\mathrm{link}_\Delta(v);\mathbb{K}) \cong \tilde{H}_j(\mathrm{del}_\Delta(v);\mathbb{K}) \tag{6.1}$$

for every j, which means that (3) holds.

Suppose Δ is not acyclic, and assume without loss of generality that Δ has the homology of a k-dimensional sphere. In other words,

$$\tilde{H}_j(\Delta;\mathbb{K}) \neq 0 \iff j = k, \quad \text{and} \quad \tilde{H}_k(\Delta;\mathbb{K}) = \mathbb{K}.$$

Theorem 2.1.7 gives us the following exact sequence

$$\begin{aligned} 0 \to \tilde{H}_k(\mathrm{link}_\Delta(v);\mathbb{K}) \to \tilde{H}_k(\mathrm{del}_\Delta(v);\mathbb{K}) \to \quad & \mathbb{K} \\ \to \quad & \tilde{H}_{k-1}(\mathrm{link}_\Delta(v);\mathbb{K}) \to \tilde{H}_{k-1}(\mathrm{del}_\Delta(v);\mathbb{K}) \to 0 \end{aligned} \tag{6.2}$$

and also (6.1) for all $j \notin \{k, k-1\}$. Since both $\mathrm{link}_\Delta(v)$ and $\mathrm{del}_\Delta(v)$ are either acyclic, or have the homology of a sphere, and the alternating sum of dimensions must add up to zero, we may simplify the sequence above to

$$\cdots \to 0 \to \tilde{H}_k(\mathrm{del}_\Delta(v);\mathbb{K}) \to \mathbb{K} \to \tilde{H}_{k-1}(\mathrm{link}_\Delta(v);\mathbb{K}) \to 0 \to \cdots$$

which implies exactly one of (1) or (2) holds. □

We are now ready to set up the required tools to find the homology of induced subcomplexes and links of spherical complexes. Some of the notation we use appeared previously in the literature for studying independence complexes of ternary graphs [109, 207].

Definition 6.3.2 ($\Delta(X \mid Y)$)**.** Given a sequence of vertices $v_1, \ldots, v_n$ of a simplicial complex Γ partitioned into disjoint sets X and Y, and a subcomplex Δ of Γ, we define the following sequence of subcomplexes.

- $\Delta_1 = \Delta$;
- if $i > 1$, then
$$\Delta_{i+1} = \begin{cases} \mathrm{link}_{\Delta_i}(v_i) & \text{if} \quad v_i \in X \\ \mathrm{del}_{\Delta_i}(v_i) & \text{if} \quad v_i \in Y. \end{cases}$$

We then denote Δ_{n+1} by $\Delta(X \mid Y)$ and similarly if

$$X_i = X \cap \{v_1, \ldots, v_i\} \quad \text{and} \quad Y_i = Y \cap \{v_1, \ldots, v_i\},$$

then we denote Δ_{i+1} by $\Delta(X_i \mid Y_i)$. In particular, $\Delta(\emptyset \mid \emptyset) = \Delta$, and for a vertex v of Γ we have

$$\Delta(\emptyset \mid v) = \mathrm{del}_\Delta(v) \quad \text{and} \quad \Delta(v \mid \emptyset) = \mathrm{link}_\Delta(v).$$

We may then define the following function analogous to the one in Kim's work [109].

Definition 6.3.3 ($d(X \mid Y)$)**.** Let Δ be a $\mathbb{K}$-spherical complex, and X, Y as in Definition 6.3.2. We set

$$d(X \mid Y) = \begin{cases} k & \text{if} \quad \tilde{H}_k(\Delta(X \mid Y); \mathbb{K}) \cong \mathbb{K} \quad \text{for some} \quad k \\ * & \text{otherwise.} \end{cases}$$

where $*$ is just a symbol.

We now define filtrations, these are the main objects we use in order to prove the main results of this chapter.

Definition 6.3.4 (**Filtrations,** [57, Definition 3.3])**.** Let Δ be a $\mathbb{K}$-spherical complex with the homology of a k-dimensional sphere, or equivalently, $d(\emptyset \mid \emptyset) = k \neq *$. A *filtration* $\mathcal{F}$ of Δ is a sequence of subcomplexes

$$\mathcal{F} := \Delta_1 \supseteq \Delta_2 \supseteq \cdots \supseteq \Delta_q$$

with a sequence of vertices $v_1 \in \Delta_1, \ldots, v_{q-1} \in \Delta_{q-1}$ such that

1. $\Delta_1 = \Delta$;
2. $\Delta_{i+1} \in \{\mathrm{del}_{\Delta_i}(v_i), \mathrm{link}_{\Delta_i}(v_i)\}$ for every i;
3. Δ_{i+1} has the homology of a sphere for every i when the coefficients are taken to be $\mathbb{K}$.

We say a filtration is *maximal* if $\Delta_q = \emptyset$.

Since each Δ_i is not acyclic, by Lemma 6.3.1 for every i, there is $t_i \in \{0, 1\}$ such that

$$\tilde{H}_j(\Delta_i; \mathbb{K}) \cong \tilde{H}_{j-t_i}(\Delta_{i+1}; \mathbb{K}) \quad \text{for all } j, \quad \text{where} \quad \Delta_{i+1} = \begin{cases} \mathrm{del}_{\Delta_i}(v_i) & \text{if} \quad t_i = 0 \\ \mathrm{link}_{\Delta_i}(v_i) & \text{if} \quad t_i = 1. \end{cases}$$

A consequence of Theorem 2.1.7 and Lemma 6.3.1 is the following.

Proposition 6.3.5 (Existence of filtrations). *Let Δ be a $\mathbb{K}$-spherical complex. If Δ is not acyclic, then it has a maximal filtration.*

Proof. Let $v_1 \in \Delta = \Delta_1$, and assume Δ has the homology of a k-dimensional sphere. By Theorem 2.1.7 we have the exact sequence in (6.2). Moreover, by the definition of spherical complex, both $\operatorname{link}_\Delta(v_1)$ and $\operatorname{del}_\Delta(v_1)$ either have trivial homology, or have the homology of a sphere. In particular, by Lemma 6.3.1 exactly one of the following holds:

1. $\operatorname{link}_\Delta(v)$ has the homology of S^{k-1} and $\operatorname{del}_\Delta(v)$ has trivial homology.

2. $\operatorname{del}_\Delta(v)$ has the homology of S^k and $\operatorname{link}_\Delta(v)$ has trivial homology.

Note that taking Δ_2 to be the complex above without trivial homology gives us the first step in a filtration. Inductively, since every simplicial complex obtained by taking links and deletions of Δ is also spherical by definition, we can repeat the process by choosing a new vertex $v_2 \in \Delta_2$. After a finite number of steps, the complex Δ_q will have only $\emptyset$ as a face, and in this case $\Delta_q = S^{-1}$, so the result holds. $\square$

Proposition 6.3.5 also implies any sequence of vertices $v_1, \ldots, v_s$ such that $\{v_1, \ldots, v_s\} \in \Delta$ can be used to build a filtration.

Lemma 6.3.6 (Analogue of Lemma 3.2, [109]**).** *Let Δ be a spherical complex and $X, Y \subseteq V(\Delta)$ such that $X \cap Y = \emptyset$. For every vertex $v \notin X \cup Y$, the triple*

$$\big(d(X \mid Y), d(X \cup \{v\} \mid Y), d(X \mid Y \cup \{v\})\big) \in \big\{(*,*,*), (k,*,k), (k+1,k,*), (*,k,k)\big\}$$

for some integer $k \geq -1$.

Proof. We split the proof in two cases.

1. If $v \notin \Delta(X \mid Y)$, then $\Delta(X \cup \{v\} \mid Y) = \{\}$ and $\Delta(X \mid Y \cup \{v\}) = \Delta(X \mid Y)$, hence

$$\big(d(X \mid Y), d(X \cup \{v\} \mid Y), d(X \mid Y \cup \{v\})\big) \in \{(*,*,*), (k,*,k)\}$$

2. If $v \in \Delta(X \mid Y)$, the result follows by Lemma 6.3.1

□

We are now ready to prove the main result of this chapter.

Theorem 6.3.7 (Bounds on homology of subcomplexes of spherical complexes, [57, Theorem 3.6]**).** *Let Δ be a spherical complex with the homology of a sphere of dimension d_Δ. Let Σ be a subcomplex of Δ such that $\Sigma = \Delta(X \mid Y)$, where X, Y are disjoint subsets of the vertex set of Δ. Then Σ either has no homology or has the homology of a sphere of dimension d_Σ, where*

$$d_\Delta - |X| - |Y| \leq d_\Sigma \leq d_\Delta.$$

Proof. Fix $\Sigma = \Delta(A \mid B)$ a subcomplex of Δ. Since Δ is a spherical complex, we know Σ either has no homology, or has the homology of a sphere of dimension d_Σ. If Σ has no homology, we have nothing to show. In the latter case, we proceed by induction on $n = |A| + |B|$ to show

$$d_\Delta - |A| - |B| \leq d_\Sigma \leq d_\Delta.$$

If $n = 1$, then either $A = \{v\}$ and $B = \emptyset$ or $A = \emptyset$ and $B = \{v\}$ for some $v \in \Delta$. The result follows directly by Lemma 6.3.6 since

$$\big(d(X \mid Y), d(X \cup \{v\} \mid Y), d(X \mid Y \cup \{v\})\big) \in \big\{(d_\Delta, *, d_\Delta), (d_\Delta, d_\Delta - 1, *)\big\}.$$

Now suppose $n > 1$ and the statement holds for every subcomplex $\Delta(A' \mid B')$ with $|A'| + |B'| \leq n$. Fix two subsets $A = \{v_1, \ldots, v_q\}$, $B = \{v_{q+1}, \ldots, v_{n+1}\}$ such that $A \cap B = \emptyset$, and let $d_\Sigma = d(A \mid B) \neq *$. By Lemma 6.3.6, for every $v_i \in A$ we have

$$(d(A \setminus \{v_i\} \mid B), d(A \mid B), d(A \setminus \{v_i\} \mid B \cup \{v_i\})) \in \{(d_\Sigma + 1, d_\Sigma, *), (*, d_\Sigma, d_\Sigma)\}.$$

If the triple is $(d_\Sigma + 1, d_\Sigma, *)$, then by the induction hypothesis applied to the complex $\Delta(A \setminus \{v_i\} \mid B)$ we have

$$d_\Delta - n \leq d_\Sigma + 1 \leq d_\Delta \implies d_\Delta - (n+1) \leq d_\Sigma \leq d_\Delta$$

and we are done.

Similarly, for every $v_i \in B$, by Lemma 6.3.6 we have

$$(d(A \mid B \setminus \{v_i\}), d(A \cup \{v_i\} \mid B \setminus \{v_i\}), d(A \mid B)) \in \{(d_\Sigma, *, d_\Sigma), (*, d_\Sigma, d_\Sigma)\}$$

If the triple is $(d_\Sigma, *, d_\Sigma)$, by the induction hypothesis applied to the complex $\Delta(A \mid B \setminus \{v_i\})$ we have

$$d_\Delta - n \leq d_\Sigma \leq d_\Delta \implies d_\Delta - (n+1) \leq d_\Sigma \leq d_\Delta$$

and we are done.

In summary, the only way for the induction step to fail, is if for every $v_i \in A$ and $v_j \in B$

$$(d(A \setminus \{v_i\} \mid B), d(A \mid B), d(A \setminus \{v_i\} \mid B \cup \{v_j\})) = (*, d_\Sigma, d_\Sigma)$$
$$(d(A \mid B \setminus \{v_j\}), d(A \cup \{v_j\} \mid B \setminus \{v_j\}), d(A \mid B)) = (*, d_\Sigma, d_\Sigma)$$

but if that is the case, then for every $C, D \subseteq A \cup B$ with $C \cup D = A \cup B$ and $C \cap D = \emptyset$ we have

$$d(C \mid D) = d(A \mid B) = d_\Sigma \neq *.$$

Moreover, by picking $u \in C$ (or $v \in D$), by setting $X = C \setminus \{u\}$ and $Y = D$ (or $X = C$ and $Y = D \setminus \{v\}$) in Lemma 6.3.6 we can conclude that

$$d(C \setminus \{u\} \mid D) = d(C \mid D \setminus \{v\}) = *.$$

Now since Δ is not acyclic, we can apply the same argument from Proposition 6.3.5 to show there is a filtration

$$\mathcal{F} : \Delta = \Delta_1 \supseteq \Delta_2 \supseteq \cdots \supseteq \Delta_n \supseteq \Delta_{n+1}$$

of Δ using the order of the vertices $v_1, \ldots, v_{n+1}$ of $A \cup B$. Note that this is possible since $d(\emptyset \mid \emptyset) \neq *$.

At the last step of the filtration we have a contradiction because for some C', D' with $C' \cup D' = (A \cup B) \setminus \{v_{n+1}\}$ and $C' \cap D' = \emptyset$, we have $d(C' \mid D') \neq *$ and $\Delta_n = \Delta(C' \mid D')$. $\square$

6.4 Applications to Leray numbers and squarefree monomial ideals

We now consider the consequences of Theorem 6.3.7 to the computation of topological (and homological) invariants of a spherical complex Δ. The first one follows directly from Theorem 6.3.7.

Definition 6.4.1. The $\mathbb{K}$-*Leray number* of a simplicial complex Δ is defined as

$$L_{\mathbb{K}}(\Delta) = \min\{d\colon \tilde{H}_d(\Sigma; \mathbb{K}) = 0 \quad \text{for all induced subcomplexes } \Sigma \subseteq \Delta\}.$$

In other words, $L_{\mathbb{K}}(\Delta)$ is the minimum d such that every induced subcomplex Σ of Δ (including Δ itself) has trivial d-homology.

Corollary 6.4.2. *Let Δ be a $\mathbb{K}$-spherical complex with $\tilde{H}_d(\Delta; \mathbb{K}) \cong \mathbb{K}$. Then*

$$L_{\mathbb{K}}(\Delta) = d + 1.$$

Proof. Let Σ be an induced subcomplex of Δ. Then

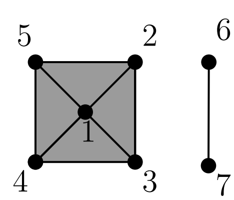
$$\Sigma = \Delta(\emptyset \mid V(\Delta) \setminus V(\Sigma)).$$

Since Δ is spherical, we know Σ is either acyclic, or has the homology of a d'-dimensional sphere. Theorem 6.3.7 then implies

$$d' \leq d,$$

so the Leray number of Δ is $d + 1$. □

Example 6.4.3. Let Δ be the following complex.



Note that the simplicial complex $\Sigma \subseteq \Delta$ induced on the set of vertices $\{2, 3, 4, 5\}$ has $\tilde{H}_1(\Sigma; \mathbb{K}) \cong \mathbb{K}$ for any field $\mathbb{K}$, while $\tilde{H}_0(\Delta; \mathbb{K}) \cong \mathbb{K}$ and $\tilde{H}_j(\Delta; \mathbb{K}) = 0$ for every $j > 0$. It can be shown that $L_{\mathbb{K}}(\Delta) = 2$, and in particular, by Corollary 6.4.2, Δ is not spherical. Indeed, the complex $\Gamma \subseteq \Delta$ induced on $\{3, 5, 6\}$ has $\tilde{H}_0(\Gamma; \mathbb{K}) \cong \mathbb{K} \oplus \mathbb{K}$.

We now define three invariants of spherical complexes: sign, depth and projective dimension. The latter two, as we will see, are equal to their algebraic counterparts, so that the names are justified. The sign is used to give a criterion for when a spherical complex is acyclic in Theorem 6.4.8.

Definition 6.4.4. Let Δ be a $\mathbb{K}$-spherical complex and $v_{i_1}, \ldots, v_{i_s}$ an ordered subset B of the vertices of Δ. We define the poset $\mathcal{G}_B$ to be the set

$$\{(C \mid D)\colon C \cup D = \{v_{i_1}, \ldots, v_{i_j}\} \subseteq B \quad \text{and} \quad C \cap D = \emptyset\}$$

with a partial order $\leq$ given by $(C' \mid D') \leq (C \mid D)$ if and only if $C \subseteq C'$ and $D \subseteq D'$. We view the poset $\mathcal{G}_B$ by its Hasse diagram, which is a binary tree.

Let $j_{\mathcal{G}_B}$ be the number of pairs $(C \mid D)$ in $\mathcal{G}_B$ such that $d(C \mid D) \neq *$ and $C \cup D = B$, and define the *sign* of $\mathcal{G}_B$ to be

$$\operatorname{sign}_{\mathcal{G}_B} = (-1)^{j_{\mathcal{G}_B}}.$$

Lemma 6.4.5. *Let Δ be a spherical complex, $B = \{v_{i_1}, \ldots, v_{i_s}\}$ an ordered set of vertices of Δ and $\mathcal{G}_B$ be a poset as in Definition 6.4.4. Let $f(t)$ denote the number of pairs $(C \mid D)$ in $\mathcal{G}_B$ such that $d(C \mid D) \neq *$ and $C \cup D = \{v_{i_1}, \ldots, v_{i_t}\}$. Then for every $0 \leq t, t' \leq |B|$ we have $f(t)$ is even if and only if $f(t')$ is even.*

Proof. Assume $(C \mid D) \in \mathcal{G}_B$ with $C \cup D = \{v_1, \ldots, v_t\}$. By Lemma 6.3.6, the possible values of the triple

$$\big(d(C \mid D),\ d(C \cup \{v_{t+1}\} \mid D),\ d(C \mid D \cup \{v_{t+1}\})\big) \tag{6.3}$$

are

$$(*, *, *), \quad (k+1, k, *), \quad (k, *, k), \quad (*, k, k)$$

for some integer $d \geq -1$.

If $d(C \mid D) \neq *$, the pair $(C \mid D)$ contributes exactly 1 pair $(C' \mid D')$ with $d(C' \mid D') \neq *$ and $C' \cup D' = \{v_{i_1}, \ldots, v_{i_{t+1}}\}$. If $d(C \mid D) = *$, the pair $(C \mid D)$ contributes 0 or 2 pairs $(C' \mid D')$ with $d(C' \mid D') \neq *$ and $C' \cup D' = \{v_{i_1}, \ldots, v_{i_{t+1}}\}$. The conclusion now follows. $\square$

Theorem 6.4.6 (**The sign of** Δ)**.** *Let* Δ *be a spherical complex,* B *a set of vertices and let* $\mathcal{G}_B$ *be a poset as in Definition 6.4.4. Then*

(1) $\operatorname{sign}_{\mathcal{G}_B}$ *is independent of the ordering of the vertices of* B*.*

(2) $\operatorname{sign}_{\mathcal{G}_B} = \operatorname{sign}_{\mathcal{G}_{B'}}$ *for any other set of vertices* B'*.*

(3) $\operatorname{sign}_{\mathcal{G}_B}$ *is* 1 *if and only if* Δ *is acyclic.*

Proof. Using (the notation in) Lemma 6.4.5, if B' is any set of vertices, with any ordering, we have

$$\operatorname{sign}_{\mathcal{G}_B} = (-1)^{f(|B|)} = (-1)^{f(0)} = (-1)^{f(|B'|)} = \operatorname{sign}_{\mathcal{G}_{B'}}$$

which settles (1) and (2). Item (3) also follows immediately, after observing that

$$f(0) = \begin{cases} 1 & \text{if} \quad d(\emptyset \mid \emptyset) \neq * \text{ (i.e. } \Delta \text{ is not acyclic)} \\ 0 & \text{if} \quad d(\emptyset \mid \emptyset) = * \text{ (i.e. } \Delta \text{ is acyclic).} \end{cases}$$

□

Definition 6.4.7. Let Δ be a non-acyclic spherical complex and a maximal filtration $\mathcal{F}$

$$\Delta = \Delta_1 \supseteq \cdots \supseteq \Delta_q = \emptyset$$

of Δ. Recalling that each step in the filtration $\mathcal{F}$ corresponds to a vertex deletion or taking a link, we will use the following notation.

1. $\operatorname{del}(\mathcal{F}) = \{v_i \colon \Delta_{i+1} = \operatorname{del}_{\Delta_i}(v_i)\}$
2. $\operatorname{link}(\mathcal{F}) = \{v_i \colon \Delta_{i+1} = \operatorname{link}_{\Delta_i}(v_i)\}$
3. The *depth* of $\mathcal{F}$ is $\operatorname{depth}(\mathcal{F}) = |\operatorname{link}(\mathcal{F})|$
4. $N(\mathcal{F}) = V(\Delta) \setminus (\bigcup_{v \in \operatorname{link}(\mathcal{F})} V(\operatorname{star}_\Delta(v)))$.

Theorem 6.4.8 (**Computing homology of a spherical complex,** [57, Theorem 4.5])**.** *Let* Δ *be a spherical complex.*

1. Δ *is acyclic if and only if* $\operatorname{sign}(\Delta) = 1$.

2. *If* $\operatorname{sign}(\Delta) = -1$ *then* Δ *has the homology of a* $(\operatorname{depth}(\mathcal{F}) - 1)$*–dimensional sphere, where* $\mathcal{F}$ *is any maximal filtration of* Δ*, and in particular* $\operatorname{depth}(\mathcal{F})$ *is an invariant of* Δ*.*

Proof. In view of Theorem 6.4.8, we only need to prove that for a non-acyclic spherical complex Δ and a maximal filtration $\mathcal{F}$ of Δ, Δ has the homology of a $(\operatorname{depth}(\mathcal{F}) - 1)$–dimensional sphere. Let $\mathcal{F}$ be a maximal filtration of Δ. By Lemma 6.3.1, each time $v_i \in \operatorname{del}(\mathcal{F})$, then

$$\tilde{H}_j(\Delta_i; \mathbb{K}) \cong \tilde{H}_j(\Delta_i; \mathbb{K}).$$

Again by Lemma 6.3.1, when $v_i \in \operatorname{link}(\mathcal{F})$ we have the following isomorphisms for all i

$$\tilde{H}_{j-1}(\Delta_{i+1}); \mathbb{K}) \cong \tilde{H}_j(\Delta_i; \mathbb{K}).$$

Finally, the empty complex $\Delta_q = \Delta(\operatorname{link}(\mathcal{F}) \mid \operatorname{del}(\mathcal{F}))$ and so Δ_q has the homology of S^{-1}. We have

$$\tilde{H}_{\operatorname{depth}(\mathcal{F})-1}(\Delta; \mathbb{K}) \cong \tilde{H}_{-1}(\Delta(\operatorname{link}(\mathcal{F}) \mid \operatorname{del}(\mathcal{F})); \mathbb{K}) \cong \mathbb{K},$$

and the result follows. □

Before stating the next lemma, we note that given a subcomplex $\Sigma = \Delta(X \mid Y) \neq \{\}$ of Δ, the set X must be a face of Δ. Indeed, given any face σ of Σ (including the only (-1)–dimensional face of Σ), we know by the definition of link that $\sigma \cup X \in \Delta$.

Lemma 6.4.9. *Let* Δ *be a non-acyclic spherical complex and* $\mathcal{F}$ *a maximal filtration of* Δ*. Then*

$$|V(\Delta)| = \operatorname{depth}(\mathcal{F}) + |\operatorname{del}(\mathcal{F})| + |N(\mathcal{F})| - |\operatorname{del}(\mathcal{F}) \cap N(\mathcal{F})|.$$

Proof. Let $(v_1, \ldots, v_s)$ be the sequence of vertices that defines the maximal filtration $\mathcal{F}$. Then by definition we have $\Delta(\operatorname{link}(\mathcal{F}) \mid \operatorname{del}(\mathcal{F})) = \emptyset$. In particular

$$V(\Delta) = \operatorname{link}(\mathcal{F}) \cup N(\mathcal{F}) \cup \operatorname{del}(\mathcal{F})$$

and since $\operatorname{link}(\mathcal{F})$ is a face of Δ, we know $\operatorname{link}(\mathcal{F}) \cap N(\mathcal{F}) = \emptyset$. We also know that $\operatorname{link}(\mathcal{F})$ is disjoint from $\operatorname{del}(\mathcal{F})$. So in the expression $|\operatorname{link}(\mathcal{F})| + |\operatorname{del}(\mathcal{F})| + |N(\mathcal{F})|$, the only vertices being counted more than once are the ones in $\operatorname{del}(\mathcal{F}) \cap N(\mathcal{F})$. □

Let Δ be a non-acyclic spherical complex Δ. In the spirit of the "Auslander-Buchsbaum formula" from commutative algebra (see Theorem 2.6.4) we define the *projective dimension* of a maximal filtration $\mathcal{F}$ of Δ as:

$$\operatorname{pdim}(\mathcal{F}) = |\operatorname{del}(\mathcal{F})| + |N(\mathcal{F})| - |\operatorname{del}(\mathcal{F}) \cap N(\mathcal{F})|.$$

Corollary 6.4.10 (Depth and projective dimension as combinatorial invariants). *Let Δ be a non-acyclic spherical complex, and let $\mathcal{F}, \mathcal{F}'$ be two maximal filtrations of Δ. Then*

1. $\operatorname{depth}(\mathcal{F}) = \operatorname{depth}(\mathcal{F}')$,

2. $\operatorname{pdim}(\mathcal{F}) = \operatorname{pdim}(\mathcal{F}')$.

Proof. This follows directly from Theorem 6.4.8 and Lemma 6.4.9. □

In light of the last results we give the following definitions, that (we will soon prove) are the combinatorial counterparts of the algebraic invariants.

Definition 6.4.11 (Depth and projective dimension of spherical complexes). Let Δ be a spherical complex. If Δ is not acyclic, let $\mathcal{F}$ be any maximal filtration of Δ. We call

$$\operatorname{pdim}(\Delta) = \operatorname{pdim}(\mathcal{F}) \quad \text{and} \quad \operatorname{depth}(\Delta) = \operatorname{depth}(\mathcal{F})$$

the *projective dimension* and *depth* of Δ, respectively.

If Δ is acyclic, we set

$$\operatorname{pdim}(\Delta) = \max\{\operatorname{pdim}(\Sigma) \colon \Sigma \text{ induced subcomplex of } \Delta \text{ and } \Sigma \text{ not acyclic}\}$$

and

$$\operatorname{depth}(\Delta) = |V(\Delta)| - \operatorname{pdim}(\Delta).$$

The following statement is a combinatorial version of the well-known "Auslander-Buchsbaum formula" (see Theorem 2.6.4) from commutative algebra, which follows now directly from Lemma 6.4.9 and Definition 6.4.11.

Theorem 6.4.12 (Auslander-Buchsbaum for spherical complexes). *Let Δ be a spherical complex. Then*

$$|V(\Delta)| = \operatorname{depth}(\Delta) + \operatorname{pdim}(\Delta).$$

We now show the relationship between the invariants of Δ defined in this section, and homological invariants of I_Δ.

Theorem 6.4.13 (Betti numbers of spherical complexes). *Let Δ be a $\mathbb{K}$-spherical complex and $\mathfrak{m}$ any squarefree monomial in $R = \mathbb{K}[x_1, \dots, x_n]$. Then the following are equivalent.*

1. $\beta_{i,\mathfrak{m}}(R/I_\Delta) \neq 0$
2. $\beta_{i,\mathfrak{m}}(R/I_\Delta) = 1$.
3. $i = \operatorname{pdim}(\Delta_\mathfrak{m})$ *and* $\Delta_\mathfrak{m}$ *is not acyclic*
4. $i = \operatorname{pdim}(\Delta_\mathfrak{m})$ *and* $\operatorname{sign}(\Delta_\mathfrak{m}) = -1$.

Proof. By definition, for every induced subcomplex $\Delta_\mathfrak{m}$ of Δ, the homology groups of $\Delta_\mathfrak{m}$ are either 0 or one (and only one) homology group is $\mathbb{K}$. By Theorem 6.4.8, $\Delta_\mathfrak{m}$ has a nonzero homology group if and only if $\operatorname{sign}(\Delta_\mathfrak{m}) = -1$, moreover, the nonzero homology group is $\tilde{H}_{\operatorname{depth}(\Delta_\mathfrak{m})-1}(\Delta_\mathfrak{m}; \mathbb{K}) \cong \mathbb{K}$. By Theorem 6.4.12 we have

$$|V(\Delta_\mathfrak{m})| = \operatorname{pdim}(\Delta_\mathfrak{m}) + \operatorname{depth}(\Delta_\mathfrak{m})$$

and so we can rewrite the isomorphism of the homology group as:

$$\tilde{H}_{|V(\Delta_\mathfrak{m})|-\operatorname{pdim}(\Delta_\mathfrak{m})-1}(\Delta_\mathfrak{m}; \mathbb{K}) \cong \mathbb{K}.$$

In particular we have:

$$\tilde{H}_{|V(\Delta_\mathfrak{m})|-i-1}(\Delta_\mathfrak{m}; \mathbb{K}) \cong \begin{cases} \mathbb{K} & \text{if } \operatorname{sign}(\Delta_\mathfrak{m}) = -1 \text{ and } \operatorname{pdim}(\Delta_\mathfrak{m}) = i \\ 0 & \text{otherwise.} \end{cases} \tag{6.4}$$

Lastly, Theorem 2.6.7 implies

$$\beta_{i,\mathfrak{m}}(R/I_\Delta) = \dim \tilde{H}_{\deg \mathfrak{m}-i-1}(\Delta_\mathfrak{m}; \mathbb{K}) = \dim \tilde{H}_{|V(\Delta_\mathfrak{m})|-i-1}(\Delta_\mathfrak{m}; \mathbb{K})$$

so the result follows by (6.4). $\square$

The following results are consequences of Theorem 6.3.7, and imply, for example, that the multigraded Betti numbers of ternary graphs behave nicely, that is, top Betti numbers come from the top.

Corollary 6.4.14. *Let Δ be a $\mathbb{K}$-spherical complex and $I_\Delta \subseteq R = \mathbb{K}[x_1, \ldots, x_n]$ its Stanley-Reisner ideal. Then*

- $\operatorname{pdim}(\Delta) = \operatorname{pdim}(R/I_\Delta)$
- $\operatorname{depth}(\Delta) = \operatorname{depth}(R/I_\Delta)$

Proof. We split the proof in two cases, when Δ is acyclic, and when it is not.

1. Assume Δ has the homology of a d_Δ-dimensional sphere and let $\Sigma = \Delta(A \mid B)$ be any induced complex of Δ such that Σ has the homology of a d_Σ-dimensional sphere. Note that for any complex $\Sigma = \Delta(A \mid B)$ we have $|V(\Sigma)| + |A| + |B| \leq |V(\Delta)|$, with equality for example if $A = \emptyset$.

   By the left hand inequality of Theorem 6.3.7 and the remark above, we have the following inequalities.

$$
\begin{aligned}
d_\Delta - |A| - |B| &\leq d_\Sigma \\
d_\Delta - d_\Sigma &\leq |A| + |B| \leq |V(\Delta)| - |V(\Sigma)| \\
|V(\Sigma)| - d_\Sigma &\leq |V(\Delta)| - d_\Delta \\
\operatorname{pdim}(\Sigma) &\leq \operatorname{pdim}(\Delta)
\end{aligned}
$$

   By Theorem 6.4.13 and the last inequality above, we see that

$$\operatorname{pdim}(R/I_\Delta) = \max(i\colon \beta_i(R/I_\Delta) \neq 0) = \max(\operatorname{pdim}(\Sigma)\colon \operatorname{sign}(\Sigma) = -1) = \operatorname{pdim}(\Delta).$$

   From the Auslander-Buchsbaum formulas

$$\operatorname{pdim}(R/I_\Delta) + \operatorname{depth}(R/I_\Delta) = \dim R = |V(\Delta)| = \operatorname{pdim}(\Delta) + \operatorname{depth}(\Delta)$$

   so the second item also holds.

2. If Δ is acyclic, then by Theorem 6.4.13 and by definition we have

$$\operatorname{pdim}(\Delta) = \max(\operatorname{pdim}(\Sigma)\colon \operatorname{sign}(\Sigma) = -1) = \max(i\colon \beta_i(R/I_\Delta) \neq 0) = \operatorname{pdim}(R/I_\Delta)$$

so the first item holds. The second item then holds by the Auslander-Buchsbaum formulas.

□

Corollary 6.4.15. *Let Δ be a non-acyclic $\mathbb{K}$-spherical complex. Then*

$$\operatorname{reg}(R/I_\Delta) = \operatorname{depth}(R/I_\Delta).$$

Proof. By Theorem 6.4.13 and the Auslander-Buchsbaum formulas we have

$$\begin{aligned}\operatorname{reg}(R/I_\Delta) &= \max(\deg \mathfrak{m} - i\colon \beta_{i,\mathfrak{m}}(R/I_\Delta) \neq 0)\\ &= \max(|V(\Sigma)| - \operatorname{pdim}(\Sigma)\colon \operatorname{sign}(\Sigma) = -1 \quad \text{and} \quad \Sigma \text{ induced})\\ &= \max(\operatorname{depth}(\Sigma)\colon \operatorname{sign}(\Sigma) = -1 \quad \text{and} \quad \Sigma \text{ induced})\end{aligned}$$

From the right hand inequality of Theorem 6.3.7, and since Δ is not acyclic we conclude

$$\operatorname{reg}(R/I_\Delta) = \max(\operatorname{depth}(\Sigma)\colon \operatorname{sign}(\Sigma) = -1) = \operatorname{depth}(\Delta) = \operatorname{depth}(R/I_\Delta).$$

□

Example 6.4.16. Let $I(G) = (x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_3x_5, x_1x_6) \subseteq S = \mathbb{K}[x_1, \dots, x_6]$ be the edge ideal of a square with two pendant leaves at nonadjacent vertices, and Δ its independence complex. The sequence

$$\Delta \supseteq \Delta(1 \mid \emptyset) \supseteq \Delta(1, 5 \mid \emptyset) = \emptyset$$

is a maximal filtration of Δ, so we conclude $\operatorname{reg}(R/I(G)) = \operatorname{depth}(R/I(G)) = 2$.

Remark 6.4.17 (Leray number and regularity). It is well known that the Leray number of Δ is related to the Castelnuovo-Mumford regularity of R/I_Δ (see [106]). More specifically,

$$L_{\mathbb{K}}(\Delta) = \mathrm{reg}(R/I_{\Delta}).$$

In particular, Corollary 6.4.15 can be seen as an algebraic analogue of Corollary 6.4.2.

6.5 Explicit examples of computations with filtrations

To showcase how filtrations could be used to simplify the search for homological invariants of graphs, we apply filtrations in Proposition 6.5.3 to recover a result about unicyclic graphs that was recently proved in [196]. In what follows, we write $G(A \mid B)$ instead of $\mathrm{Ind}(G)(A \mid B)$. The statement we prove will assume the graph is ternary as it is most closely related to the theme of this chapter, but the same argument can be made to prove the non-ternary case. To do so we need the following statements.

Lemma 6.5.1. *Let G be a ternary graph G with two connected components H_1 and H_2. Then $\mathrm{Ind}(G)$ is not contractible if and only if $\mathrm{Ind}(H_1)$ and $\mathrm{Ind}(H_2)$ are not contractible. In this case, $\mathrm{Ind}(G) \cong S^{n+m+1}$ where $\mathrm{Ind}(H_1) \cong S^n$ and $\mathrm{Ind}(H_2) \cong S^m$.*

Proof. The proof follows directly from the fact that the join of two spheres is a sphere (see, e.g. [80, Exercise 0.18]), and Theorem 6.2.1. □

Lemma 6.5.2 ([113, Proposition 4.6])**.** *The independence complex of the path graph G on n vertices satisfies the following equalities.*

$$\mathrm{Ind}(G) \cong \begin{cases} S^{k-1} & \text{if } n = 3k \\ \text{a point} & \text{if } n = 3k+1 \\ S^{k} & \text{if } n = 3k+2 \end{cases}$$

Proposition 6.5.3. *Let $G_{n,m}$ be a unicyclic graph as in Figure 6.1 where $n \geq 3$, $m \geq 1$, $3 \nmid n$, and $n \neq m$ mod 3. Then*

$$\mathrm{depth}(R/I(G_{n,m}) = \mathrm{reg}(R/I(G_{n,m})) = \begin{cases} \frac{n-1}{3} + \frac{m}{3} & n = 1,\, m = 0 \textit{ mod } 3 \\ \frac{n-1}{3} + \frac{m-2}{3} + 1 & n = 1,\, m = 2 \textit{ mod } 3 \\ \frac{n-2}{3} + \frac{m}{3} + 1 & n = 2,\, m = 0 \textit{ mod } 3 \\ \frac{n-2}{3} + \frac{m-1}{3} + 1 & n = 2,\, m = 1 \textit{ mod } 3. \end{cases}$$

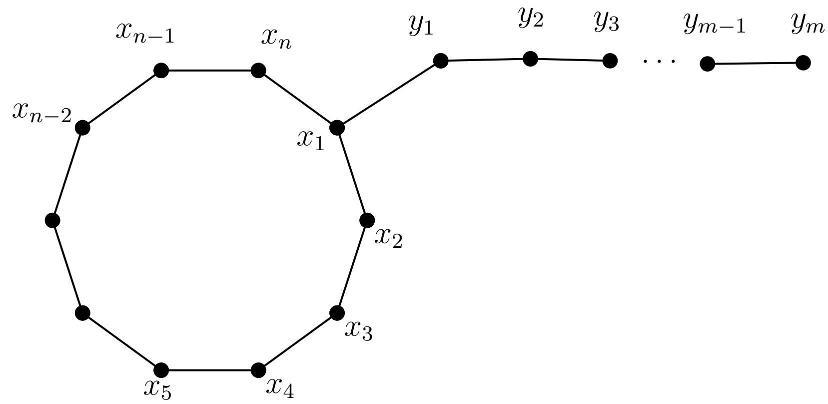


Figure 6.1: The unicyclic graph in Proposition 6.5.3

Proof. Note that $G_{n,m}(x_1 \mid \emptyset)$ and $G_{n,m}(\emptyset \mid x_1)$ are both disconnected graphs where every connected component is a path graph. By Lemma 6.5.2, if a connected component of a ternary graph is a path graph on $3k+1$ vertices, its independence complex is contractible.

1. If $n = 3k+1$ and $m = 3l$, then the two connected components of $G_{n,m}(x_1 \mid \emptyset)$ have $3(k-1)+1$ and $m-1$ vertices respectively, so $d(x_1 \mid \emptyset) = *$ by Lemmas 6.5.1 and 6.5.2. On the other hand, $G_{n,m}(\emptyset \mid x_1)$ has two connected components that are paths with $3k$ and $3l$ vertices respectively. By Lemma 6.5.2, we conclude $\mathrm{Ind}(G_{n,m}(\emptyset \mid x_1)) \cong S^{k+l-1}$ and by Lemma 6.3.6 we conclude $\mathrm{Ind}(G_{n,m}) \cong S^{k+l-1}$.

2. If $n = 3k+1$ and $m = 3l+2$, then just as in the previous case $d(x_1 \mid \emptyset) = *$ and $d(\emptyset \mid x_1) \neq *$, so $d(\emptyset \mid \emptyset) \neq *$. Since $m = 3l+2$ and $G_{n,m}(\emptyset \mid x_1)$ has one connected component with $3k$, and one with $3l+2$ vertices, by Lemmas 6.5.1 and 6.5.2 we conclude $d(\emptyset \mid x_1) = k+l$ and by Lemma 6.3.6, we conclude $\mathrm{Ind}(G_{n,m}) \cong S^{k+l}$.

By similar arguments, if $n = 3k+2$ and $m = 3k$, we conclude $\mathrm{Ind}(G_{n,m}) \cong S^{k+l}$ and if $n = 3k+2$ and $m = 3k+1$, $\mathrm{Ind}(G_{n,m}) \cong S^{k+l}$.

In all the cases above, Theorem 6.4.8, Corollaries 6.4.14 and 6.4.15 imply $\mathrm{depth}(R/I(G)) = \mathrm{reg}(R/I(G))$. The computations above together with Theorem 6.4.8 and Corollary 6.4.14 then imply the equalities as desired. $\square$

Chapter 7

Conclusion

"And when I'm back in Chicago, I feel it
Another version of me, I was in it
I wave goodbye to the end of beginning"

DJO – *"The end of the beginning"*

Throughout this thesis we considered several problems in combinatorics and commutative algebra, mostly motivated by the study of f-vectors of simplicial complexes. In this final chapter, we consider questions that follow from our work, and mention related work by the author.

7.1 Unexpectedness

7.1.1 Unexpected algebraic g-conjecture

In Chapter 3 we introduced "unexpected systems of parameters" (from [94]). These sequences of polynomials arise by studying the failure of Lefschetz properties of monomial algebras defined by ideals of the form $I_\Delta + (x_1^a, \ldots, x_n^a)$.

In combinatorics, it is often the fact that an algebra *satisfies* some Lefschetz-type property (e.g. the WLP) that is relevant, as it implies unimodality of some sequence. The definition of unexpected s.o.p.s, however, is based on the idea from (algebraic and differential) geometry that *failure* of a Lefschetz-type property is often explained by some interesting geometric phenomenon (e.g. counterexamples to Hesse's claims, unexpected hypersurfaces and Laplace equations).

Considering the results shown in Chapter 3, we can make explicit parallels between "failure" of WLP of monomial algebras and "interesting phenomena". In fact, "every" currently known example of unexpected s.o.p. was studied before in a completely different setting, with combinatorial motivations:

1. The colored s.o.p. of balanced homology spheres.

2. "The" universal s.o.p. of a homology sphere.

The combinatorial study of both settings mentioned above has (surprisingly) many similarities. A specific commonality between both scenarios that is relevant to us, is the study of their Lefschetz properties.

On one hand, the universal s.o.p. of the Stanley-Reisner ideal of the boundary of a simplex is the ideal $(e_1, \dots, e_n)$ generated by the elementary symmetric polynomials, and it is well-known that it defines the cohomology ring of the flag variety, one of the most well-studied varieties from a combinatorial perspective. In particular, the fact that $(e_1, \dots, e_n)$ defines an algebra that satisfies the SLP is a very important statement in algebraic combinatorics (although it often goes unnoticed).

On the other hand, the study of Lefschetz properties of the colored s.o.p. of balanced homology spheres were considered before for example in [101, 156]. The purpose of considering Lefschetz properties of colored s.o.p.s is that they induce finer gradings that can then be used to show special inequalities that the h-vectors of balanced spheres must satisfy, but general spheres do not necessarily satisfy. Although such inequalities have been shown before, the system of parameters used is often a slight variation of the colored system of parameters we considered in Section 3.5. The same s.o.p. we considered was studied for example in [34], where it is conjectured that this s.o.p. behaves well with Lefschetz properties.

Taking into consideration the relationship between the universal s.o.p. and the flag variety, as well as [34, Conjecture 1.3] (and computational evidence), we ask the following.

Question 7.1.1 (**Unexpected algebraic g-conjecture,** [94, Question 7.1]). *Let Δ be a d-dimensional $\mathbb{Q}$-homology sphere on vertex set $[n]$, and $\theta_1, \dots, \theta_{d+1}$ an a-unexpected s.o.p. of Δ. Does the algebra*

$$\frac{\mathbb{C}[x_1, \dots, x_n]}{I_\Delta + (\theta_1, \dots, \theta_{d+1})}$$

satisfy the SLP?

A special case of Question 7.1.1 is when the unexpected s.o.p. is the ideal generated by elementary symmetric polynomial. In this case, Question 7.1.1 can be thought of as a "coinvariant" version of the algebraic g-conjecture (see [93]).

In joint ongoing work with Mitsuki Hanada, we are considering the coinvariant case of Question 7.1.1 for low dimensional spheres ($d \leq 2$). A proof for the $d = 1$ follows from the notion of higher Hessians (see for example [78, Section 3.6])[1].

7.1.2 Is it really "the" universal system of parameters?

In Example 3.6.2 we recalled the definition of "the" universal s.o.p., which was introduced (with this name) by Herzog and Moradi [85]. Later, in Example 4.3.9 we provided computational evidence that the s.o.p. that arises from elementary symmetric polynomials is only one of many s.o.p.s with this property, suggesting that "universality" is a property that a s.o.p. can have, and it is not unique to elementary symmetric polynomials. We now pose a concrete question about universal s.o.p.s, and we note that we are able to identify different polynomials in different polynomial rings as "the" elementary symmetric polynomials, since they are all restrictions of the elementary symmetric function (i.e. a symmetric polynomial on infinitely many variables). This distinction is necessary, as it allows us to identify symmetric polynomials over different polynomial rings.

Question 7.1.2. *Let (n, a) be a tuple such that the algebra*[2]

$$A = \frac{\mathbb{Q}[x_1, \ldots, x_n]}{(x_1 \ldots x_n, x_1^a, \ldots, x_n^a)}$$

fails the WLP due to surjectivity in degree $t - 1$ for some t. Then there exists a polynomial F of degree t such that $F \in \ker \times L^\top : A_t \to A_{t-1}$, and

1. *F is the Macaulay dual generator of an algebra defined by symmetric polynomials $g_1, \ldots, g_n$, where $g_1 = e_1$ and $g_n = x_1 \ldots x_n = e_n$.*

2. *The ideal $I_\Delta + (g_1, \ldots, g_{n-1})$ is artinian for any $(n-2)$–dimensional complex Δ, and the g_i are the restriction of the corresponding symmetric function in the polynomial ring containing I_Δ. In particular, $g_1, \ldots, g_{n-1}$ form a universal s.o.p. for squarefree monomial ideals.*

3. *The polynomial F is anti-symmetric, and satisfies the assumptions of Theorem 4.3.4.*

[1] In the case $d = 1$ we are also able to show that certain Hessians satisfy the correct signature pattern in order to conclude the Hodge-Riemann relations in certain degrees. The proof follows by computing the eigenvalues of a certain tridiagonal matrix.

[2] It is also natural to ask whether we can replace $\mathbb{Q}$ by an arbitrary field $\mathbb{K}$.

Remark 7.1.3 (Conca-Krattenthaller-Watanabe-type conjectures, [30]**).** In 2009, Conca, Krattenthaller and Watanabe considered the question of when does a set of certain symmetric polynomials (complete symmetric polynomials and powers sums) form a regular sequence. Question 7.1.2 can be seen as a variation of the questions studied in [30], where we ask for collections of symmetric polynomials that form a regular sequence, but we require that e_1 and e_n are contained in the set, while allowing the other symmetric polynomials to have any form.

7.2 Probabilistic failure

Perhaps the most surprising application of Chapters 3 and 4 is that there exists settings where failure of WLP should be *expected*. There are several natural questions that arise from our results. We now list some of these questions. It should be noted that we believe new methods are needed to solve most (if not all) of the questions listed in this section.

7.2.1 Inside vs outside the interval $(c_d, d+1)$

The strategy for proving the main statements of Chapter 4 can be explained simply in two steps, each of one a statement that ends up being of its own interest, and requires different techniques to prove. The steps are:

1. Prove that top homology causes a map to fail to be surjective
2. Prove that a simple inequality of f-vectors causes the Hilbert function of an algebra to be decreasing at the spot above.

These two steps combined show that a map that cannot be injective is also not surjective, hence failure of the WLP happens.

By using methods from the probabilistic study of simplicial homology, we are able to show that there exists an open interval (c_d, ∞) such that if c is in this interval, and the probability parameter in the Linial-Meshulam model is $\frac{c}{n}$, the probability of the map failing to be surjective converges to 1 as n goes to infinity. Using well-known estimates from probability theory on the probability mass function of a binomial random variable, we are able to show that if $c < d+1$, and the probability parameter is $\frac{c}{n}$, the probability that the Hilbert series of the algebra is decreasing at the correct spot converges to 1 as n goes to infinity. In summary, we find an interval $(c_d, d+1)$ such that if c is in the interval and the probability parameter is $\frac{c}{n}$, we may conclude failure of WLP with high probability in the settings mentioned in Chapter 4.

The first natural question is what is the size of the interval $(c_d, d+1)$. Although it is never empty, as we mentioned in Chapter 4, it is known that $\lim_{d\to\infty} c_d = d+1$. This says that the size of the interval converges to 0 as the dimension d goes to infinity. To give an idea of how fast the size of the interval goes to 0, we compute c_d for low values of d below.

d	2	3	4	5	6	7	8
c_d	2.783	3.91	4.962	5.984	6.993	7.997	8.998

In particular, for $d = 8$ the size of the interval is already at the scale of 0.02.

The computation above might suggest that the interval $(c_d, d+1)$ is a very special interval. We, however, ask the following bold question

Question 7.2.1. *Can the interval $(c_d, d+1)$ be replaced by $(0, \infty)$ in Theorem 4.4.7? Moreover, can the $d+2$ be replaced by any number a where the monomial almost complete intersection of the form $(x_1 \dots x_n) + (x_1^a, \dots x_n^a)$ fails the WLP due to surjectivity? (for $(n-2)$–dimensional complexes)*

Perhaps a more reasonable question is whether the interval can be replaced by (c_d, ∞).

Finally, we note that Nagel and Petrović [155] explored the case where the probabilistic model is the algebraic version of the Erdős-Rényi model for random graphs. The models they consider directly output artinian ideals, and have no requirement on the generators that are not pure powers to be squarefree. Surprisingly, their conclusions are almost opposite to the ones we found. In particular, [155, Conjecture 3.9] as well as the computations they present, suggest that failure is a very rare phenomenon for their model. At the moment, it is completely unclear whether there is a structural difference between the models, whether there is a concrete difference between the types of ideals considered, or if this discrepancy can be explained with some other method.

7.2.2 Beyond homology: Lefschetz properties and multigraded Betti numbers

One aspect of the study of Lefschetz properties that we have not mentioned so far, is the search for structural properties of free resolutions that force failure of the WLP (or SLP). This approach to studying failure has been considered for example by Schenck and Grate in [72], as well as Abdallah and Schenck in [1]. One way of interpreting some of our results, is that the homology of a simplicial complex together with a (not very restrictive) inequality on the f-vector, causes an algebra to fail the WLP. By Hochster's formula, a complex Δ having homology is equivalent to asking that a certain multigraded Betti number is nonzero in the free resolution of I_Δ. It is then natural to ask whether a similar criterion could be shown for monomial ideals that do not contain pure powers of the variables. More concretely, we ask the following.

Question 7.2.2. *Let $I \subset R = \mathbb{K}[x_1, \dots, x_n]$ be a monomial ideal such that $x_k^a \notin I$ for any k, a and m is the lcm of all the generators of I. Moreover, let $J = I + (x_1^b, \dots, x_n^b)$. Is there an integer i such that if $\beta_{i,m}(I) \neq 0$, then a multiplication map by $L = x_1 + \cdots + x_n$ is not surjective in R/J in degree t? In other words, can we guarantee failure of surjectivity of the map $\times L : (R/J)_{t-1} \to (R/J)_t$ (for some t) [3] based on the (multigraded) Betti table of I?*

More generally, can one use the existence of a syzygy to generate a polynomial causing failure of surjectivity?

Question 7.2.2 could also suggest that there is a subtle relationship between polarization and the WLP. We note that Adam Lucas [124] studied a different question regarding WLP and polarization in his Master's thesis, but no connection between polarization and the WLP was found.

Finally, we note that the reverse relationship is also of interest. In Section 3.7 we studied a setting where we can guarantee that maps are indeed surjective. Our proof is based on the notion from combinatorial topology known as collapsibility [4]. This property is well-known to be stronger than a complex being acyclic. In fact, collapsibility should be compared to acyclicity (at least in top homological degree) just as a matrix being upper triangular

[3] Note that this is not equivalent to saying failure of WLP, as the map could still be injective

[4] As is mentioned in Chapter 3, the results could be stated using the notion of "d-collapsibility", which is weaker than collapsibility.

should be compared to a matrix having full rank. In particular, we ask the following broad question.

Question 7.2.3. *If a d-dimensional complex Δ has $\tilde{H}_d(\Delta;\mathbb{K}) = 0$, can we guarantee a multiplication map is surjective? More concretely, do the results in Section 3.7 hold if we replace the collapsible assumptions by a complex being acyclic?*

7.3 Roller Coasters

In Chapter 5 we explored how one can use results from areas of combinatorics such as graph theory, in order to generate examples that are not very intuitive from a purely algebraic perspective. More specifically, we translated the techniques used in the proof of the "Roller Coaster" conjecture about independence polynomials of well-covered graphs, to algebra, and in doing so, we show that the first half of the h-vectors of artinian Gorenstein algebras are unconstrained. In order to understand the motivation for the questions we pose in this section, we recall that the Roller Coaster conjecture was stated after counterexamples to previous conjectures (about unimodality) were found. In particular, the Roller Coaster conjecture (which is now a theorem) can be seen as the output of a series of conjectures (about unimodality) that turned out to be false. At the moment, some of the conjectures about unimodality are still open, namely:

1. ([5]) The independence polynomial of a tree is unimodal [5].

2. ([120]) The independence polynomial of a *very well-covered graph* is unimodal.

The largest known class of very well-covered graphs is the class of *whiskered graphs*. It is well-known that such graphs have Cohen-Macaulay edge ideals[6]. In view of the results by D'Alí and Venturello [38] that relate Koszul algebras with Cohen-Macaulay edge ideals, we ask the following.

[5]We note that even though this is still open, for a while it was believed that the independence polynomial of a tree should have *log-concave* coefficients. In [205] computation was done in order to verify log-concavity up to 20 vertices, but a counterexample to log-concavity on 26 vertices was later found in [102].

[6]The earliest proof of this fact is due to Villarreal [197], where the author notices that whiskering is the same as polarizing the ideal $I(G) + (x_1^2, \dots, x_n^2)$, which is artinian and hence Cohen-Macaulay.

Question 7.3.1. *Let $d \gg 0$ be an integer and π a permutation of the numbers $\{1, \ldots, \lfloor d/2 \rfloor\}$. Is there an artinian Gorenstein Koszul algebra A with Hilbert series $\sum_{i=0}^{d} h_i t^i$ such that*

$$h_{\pi(1)} < \cdots < h_{\pi(\lfloor d/2 \rfloor)}?$$

In other words, is the collection of first halves of h-vectors of artinian Gorenstein Koszul algebras unconstrained?

We note that if there is a "Roller Coaster theorem" for very well-covered graphs, then it would suggest (based on the proof of Theorem 5.5.1) that Question 7.3.1 has a positive answer. Notice however that both the graph theory conjecture and Question 7.3.1 could both have affirmative answers, and that would not be a contradiction, as the Koszul Roller Coaster algebras could arise from other settings.

7.4 Spherical complexes

In Chapter 6 we introduced a new class of simplicial complexes called "spherical complexes", which are motivated by independence complexes of ternary graphs and independence complexes of simplicial trees. These complexes share properties with homology spheres, while being less well-behaved in general. Moreover, these two classes of complexes intersect, but neither of them is contained in the other.

In [57] the authors asked whether a certain set of conditions was enough to characterize Gorenstein spherical complexes that are not cones (i.e. spherical homology spheres), see [57, Question 6.5]. A positive answer to [57, Question 6.5] can be found in [10], where the authors show the following.

Theorem 7.4.1 ([10, Theorem 3.4][7])**.** *Let Δ be a d-dimensional $\mathbb{Q}$-spherical complex that is not a cone. Then Δ is a $\mathbb{Q}$-homology sphere if and only if Δ has the homology (over $\mathbb{Q}$) of a d-dimensional sphere.*

More generally, in [10] the authors show that if Δ is a flag homology sphere and Δ is the independence complex of a planar graph G, then Δ is vertex decomposable and Δ is polytopal. We then ask the following question, which greatly strenghtens Theorem 7.4.1.

[7] Although the statement is only for ternary graphs, the arguments generalize directly to $\mathbb{Q}$-homology spheres, see [10, Remark 3.5].

Question 7.4.2. *Let Δ be a $\mathbb{Q}$-spherical complex that is not a cone. Is it true that Δ is the boundary complex of a vertex decomposable simplicial polytope?*

Finally, we note that every link (and deletion) of a spherical complex either has the homology of a sphere, or is acyclic, the euler characteristic of every link is very close to having the same property as the euler characteristics of links of spheres. With this connection in mind, we ask the following broad question.

Question 7.4.3. *What are inequalities that the f-vector of any (flag, nonacyclic) spherical complex must satisfy?*

Although one could hope that the f-vector of any spherical complex is unimodal, results in graph theory can be used to show that this is not the case. More concretely, due to the conjectural unimodality of the independence polynomial of trees [5], it was once asked whether the independence polynomial of any bipartite graph is unimodal [119].

Let $K_{59,100}$ be the complete bipartite graph with bipartition $V_1 \cup V_2$ where $|V_1| = 59$ and $|V_2| = 100$. Let G be the graph obtained from $K_{59,100}$ by adding pendant leaves to every vertex of V_2. In 2013, Bhattacharyya and Kahn [11] showed that the independence polynomial of G is not unimodal. In particular, disproving the conjecture by Levit and Mandrescu from 2006 [119].

It is easy to see that for every $x, y \in V_1$ and $w, z \in V_2$, the induced subgraph of G on x, y, z, w contains a square. In particular, the graph G from [11] is a ternary graph with a nonunimodal independence polynomial. We then conclude that the f-vector of a spherical complex does not need to be unimodal.

7.5 Final remarks

Throughout this thesis we explored several problems arising in commutative algebra, combinatorics and topology. Our motivation for studying most of the problems came from the simple question of understanding sequences of numbers (e.g. f-vectors) that arise from certain objects (e.g. special classes of simplicial complexes). By now it should be clear that these questions are only simple at first glance, and in fact often hide a lot of structure.

Most of the results we obtain can be viewed as algebraic middle steps towards answering such questions. A concrete example (and perhaps the most important example) can be seen in our results in Chapters 3 and 4, which are inherently algebraic, but allow us to ask Question 7.1.1, which if true, could have interesting consequences to the theory of f-vectors of special classes of spheres. An example of the types of consequences can be found for example in the inequalities obtained in [156] for balanced spheres, which uses a variation of the colored s.o.p. introduced by Stanley.

From an algebraic/geometric perspective, our results in Chapter 3 put the study of Lefschetz properties of monomial ideals in a very similar framework as the geometric study of Lefschetz properties. Up until now, there was no combinatorial meaning to failure of the WLP in large settings. We note however that geometric explanations for the failure of WLP of monomial algebras can be find throughout the literature, for example in the context of Togliatti systems and Laplace equations [142].

Our results in Chapter 4 show that the intuition most people in the area have does not hold for the class of algebras defined by ideals of the form $I_\Delta + (x_1^a, \ldots, x_n^a)$. We obtain such results by applying techniques from the probabilistic study of simplicial homology, and we believe there is a lot more that can be done: improving the exponents allowed, increasing the size of the interval, etc. One of the biggest takeaways from the techniques introduced in Chapter 4 is that there is indeed a connection between failure of WLP and homology. Our hope is that this could lead to a more general relation between syzygies (or Betti tables) and failure of the WLP. We believe a special interesting direction is the relationship between the regularity of I_Δ and the WLP of algebras defined by $I_\Delta + (x_1^a, \ldots, x_n^a)$.

In Chapter 5, we show that the history of the study of h-vectors of Gorenstein algebras is very similar to the history of the study of independence polynomials of graphs. Both areas have several (now disproven) conjectures suggesting “good behaviour”, and many results showing that the “extreme” cases can be arbitrarily bad. We believe the important intuition that can be taken from such similarities is that results about independence polynomials of graphs serve as “lower bounds” to how bad the behaviour of h-vectors of Koszul algebras can be. Moreover, it is clear that several questions that have been and are currently being studied from the graph theory perspective are very interesting questions from the algebraic perspective as well. As a concrete example, the study of which permutations can be realized by reordering the h-vector of a (Gorenstein) Koszul algebra (and what is the smallest codimension of such an algebra).

Finally in Chapter 6 we introduced a new class of simplicial complexes which are motivated by the class of ternary graphs, and the topological properties satisfied by their independence complexes. Although at first this topic might seem completely different from the others, we note that these complexes satisfy similar properties to homology spheres (but are in general less well-behaved) and so our hope is that by putting the independence complexes of trees in a similar setting to homology spheres, insights from the theory of f-vectors of spheres can be adapted in order to obtain new results about the independence polynomials of trees.

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Appendix A

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