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The log-concavity of chromatic polynomials

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por

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Resumo

Este trabalho é baseado no artigo *Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs*, de J. Huh, publicado no *Journal of the American Mathematical Society* (2012), tem como objetivo estudar os métodos utilizados por esse autor para provar um caso particular da conjectura de Rota-Heron-Welsh sobre a log-concavidade dos coeficientes dos polinômios característicos de um matroide. J. Huh utilizou técnicas de topologia algébrica, combinatória, geometria algébrica e álgebra comutativa para provar que os coeficientes do polinômio característico de um matroide representável sobre um corpo de característica zero formam uma sequência log-côncava.

Palavras-chave: Matroides; log-concavidade; polinômio cromático; multiplicidade mista.

Abstract

This work is based on the article *Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs*, by J. Huh, published in Journal of the American Mathematical Society (2012). The dissertation is devoted to the study of the methods used by Huh to prove a particular case of the Rota-Heron-Welsh conjecture. J. Huh used results from algebraic topology, combinatorics, algebraic geometry and commutative algebra to prove the log-concavity of the coefficients of the characteristic polynomial of a matroid representable over a field of characteristic zero.

Keywords: Matroids; log-concavity; chromatic polynomial; mixed multiplicities.

Contents

Introduction

The concept of chromatic polynomial for planar graphs was first introduced in 1912 by G. Birkhoff in an attempt to solve the famous four color problem. It was later generalized to all graphs by H. Whitney in 1932. Since then, many properties of the chromatic polynomial have been studied.

In 1968, R. Read conjectured that the coefficients of the chromatic polynomial of a graph form a unimodal sequence. Let G be a graph and let its chromatic polynomial be $\chi_G(x) = c_0x^n + \cdots + c_ix^{n-i} + \cdots + c_n$.

Conjecture 1 (Read)**.** [Unimodality for graphs] The coefficients (of the chromatic polynomial of a graph) first increase in absolute magnitude, and then decrease; two successive coefficients may be equal, but it seems that there is never one coefficient flanked by larger coefficients.

Later, in 1974, S. Hoggar conjectured that the coefficients of chromatic polynomials satisfy log-concavity, which is stronger than unimodality.

Conjecture 2 (Hoggar)**.** [Log-concavity for graphs] Let G be a graph and $\chi_G(x) = c_0x^n + \cdots + c_ix^{n-i} + \cdots + c_n$ its chromatic polynomial. Then the sequence $c_0, \ldots, c_n$ is a sign-alternating log-concave sequence with no internal zeros.

In his paper [12], the author also mentions that the conjecture has been verified for graphs with less than 7 vertices.

In 1976, Welsh conjectured a more general version of the conjecture above:

Conjecture 3 (Rota-Welsh)**.** [Log-concavity for matroids] Let M be a matroid. Then the coefficients of the characteristic polynomial of M form a sign-alternating log-concave sequence with no internal zeros.

Many years later, in 2006 [17], the authors mention that conjecture 2 had been verified for graphs with up to 11 vertices, and that there had been close to no development towards solving conjectures 1 and 2.

It is also well known that conjecture 1 is weaker than conjecture 2 which is weaker than conjecture 3. In particular, if the matroid M in conjecture 3 is assumed to be representable over $\mathbb{C}$, then the conjecture still implies conjecture 2.

One important aspect of the conjectures above comes from the well-known *deletion-contraction* lemma:

Lemma 1. The chromatic polynomial of a graph is the sum of two chromatic polynomials of graphs.

From the lemma above, the conjectures imply that the sum of some log-concave sequences is still log-concave (when the sequences correspond to the coefficients of chromatic polynomials of the deletion and contraction of an edge of a graph), which is not true in general.

In this text we outline some details of the recent proof of a particular case of conjecture 3, where the matroid M is assumed to be representable over a field of characteristic 0.

Chapter 1 includes the required preliminaries from commutative algebra for the rest of the text. Chapter 2 includes the commutative algebra machinery required for the proof.

Chapter 3 includes a brief introduction to algebraic topology and one of the main theorems of [13]. A proof of theorem 3.3.1 is given in appendix A.

In chapter 4 and 5 the main combinatorial objects in the conjectures are defined. Chapter 5 also includes examples of unimodal sequences in commutative algebra.

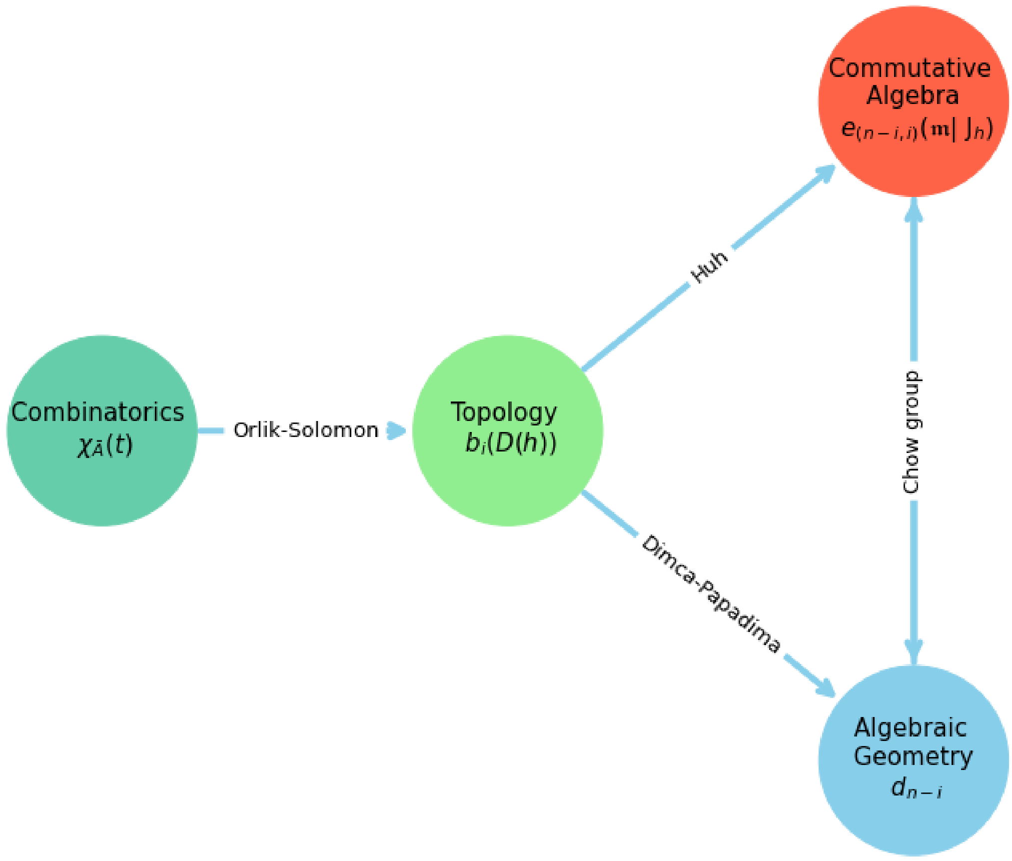


Figure 1: A visualization of the connections between the areas involved in the proof.

In chapter 6 we define the chow group of $\mathbb{P}^n \times \mathbb{P}^m$ and define the last invariant needed to state the most important theorem in the proof of conjecture 2.

Chapter 7 includes the last details needed to finish the proof of the conjectures and a brief timeline of the proof of the conjectures. As of today, even conjecture 3 has been proven (see [13], [15] and [16]).

Finally, a simple application of (mixed) multiplicities to biomathematics can be found in appendix B.

Chapter 1

Preliminaries

In this chapter we define the basic objects and state the results we will use later on. We first set some basic definitions and notation that will be used throughout this text.

Definition 1.0.1. Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over an arbitrary field k. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. We denote by x^α the monomial $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. Clearly $x^\alpha x^\beta = x^{\alpha+\beta}$ for every $\alpha, \beta \in \mathbb{N}^n$.

Definition 1.0.2. Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over an arbitrary field k. We say an ideal $I \subset R$ is a monomial ideal if it is generated by monomials.

Definition 1.0.3. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a point in $\mathbb{N}^n$ for some $n > 0$. We denote by $\alpha!$ the product $\alpha_1! \ldots \alpha_n!$, and by $|\alpha|$ the sum $\alpha_1 + \cdots + \alpha_n$.

If a proof is ommited in this chapter, a reference to where the reader may find the complete proof will be mentioned.

1.1 Length

Let R be a Noetherian ring and M a R-module. A chain of submodules of M of size n is a sequence of strict inclusions

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M.$$

A *composition series* of M is a maximal chain, that is, a chain such that no more submodules can be inserted.

Proposition 1.1.1. Following the above notation, suppose M has a composition series of size n. Then every composition series of M has size n, and every chain can be extended to a composition series.

Proof. See [1, p. 77 Proposition 6.7]. □

If the size of a composition series of a module M is finite, we say M is a module *of finite length*. The number $\ell(M)$ will denote the size of a composition series of M and is called the *length* of M. The notion of length generalizes the concept of dimension in linear algebra to R-modules.

In this section we will state basic results that will be used in the following sections.

Proposition 1.1.2. Let R be a ring and M a R-module. The module M has finite length if and only if it is both Noetherian and Artinian.

Proof. See [1, p. 77 Proposition 6.8]. □

Proposition 1.1.3. Let R be a Noetherian (resp. Artinian) ring and M a finitely generated R-module. Then M is Noetherian (resp. Artinian).

Proof. See [1, p. 76 Proposition 6.5]. □

Proposition 1.1.4. Let R be a ring. The following are equivalent:

- R is Artinian.
- R is Noetherian and $\dim R = 0$.

Proof. See [1, p. 90 Theorem 8.5]. $\square$

From these three propositions we can conclude the following fact that is very useful:

Corollary 1.1.5. Let R be an Artinian ring and M a finitely generated R-module. Then M has finite length.

Lastly we will need the fact that length of modules is additive on short exact sequences:

Proposition 1.1.6. Let M', M and M'' be R-modules such that there is a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

and M has finite length. Then $\ell(M') < \infty$, $\ell(M'') < \infty$ and $\ell(M) = \ell(M') + \ell(M'')$.

Proof. See [1, p. 75 Proposition 6.3] and [1, p. 77 Theorem 6.9]. $\square$

1.2 Associated primes

Definition 1.2.1. Let R be a ring and M an R-module. A prime ideal $\mathfrak{p}$ of R is called an associated prime ideal of M if $\mathfrak{p} = (0\colon x)$ for some $x \in M$, that is, $\mathfrak{p} = \operatorname{ann}(x)$. The set of associated primes of M will be denoted by $\operatorname{Ass}_R(M)$.

Definition 1.2.2. Let R be a ring and I a proper ideal. We say I is primary if

$$xy \in I \implies x \in I \text{ or } y^n \in I \text{ for some } n \in \mathbb{N}$$

Or equivalenty,

$$I \text{ is primary} \iff A/I \neq 0 \text{ and every zero-divisor in } A/I \text{ is nilpotent}$$

Moreover, if I is primary and $\sqrt{I} = \mathfrak{p}$ we say I is $\mathfrak{p}$-primary.

Example 1.2.3. Let $G = (V, E)$ be a simple graph where $V = (1, \ldots, n)$. Let k be an arbitrary field and $R = k[x_1, \ldots, x_n]$. Define $I(G)$ to be the ideal generated by $\{x_i x_j \mid (i, j) \in E\}$. Then the associated primes of $I(G)$ are exactly prime ideals of the form $(x_{i_1}, \ldots, x_{i_r})$ where $\{i_1, \ldots, i_r\}$ is a minimal vertex cover [1] of G. For a proof see [25, p. 168 Proposition 6.1.16].

Example 1.2.4. Let k be an arbitrary field and $R = k[X, Y, Z]/(XY - Z^2)$. Let x, y, z denote the images of X, Y, Z in R. Let $\mathfrak{p} = (x, z)$, the ideal (X, Z) is a prime ideal of $k[X, Y, Z]$ that contains $(XY - Z^2)$ so we conclude $\mathfrak{p} = (X, Z)/(XY - Z^2)$ is also a prime ideal. Note that $z^2 = xy \in \mathfrak{p}^2$ but $x \notin \mathfrak{p}^2$ and $y^n \notin \mathfrak{p}^2$ for any $n > 0$. These remarks prove that $\mathfrak{p}^2$ is not primary, even though $\sqrt{\mathfrak{p}^2} = \mathfrak{p}$. This example shows that in general primary ideals are not as simple as in $\mathbb{Z}$, it is possible for powers of primes to not be primary.

Later on we will need the following propositions:

Proposition 1.2.1. Let R be a Noetherian ring and M a nonzero module. Then

1. $\operatorname{Ass}(M) \neq \emptyset$.

[1] A vertex cover T of a graph $G = (V, E)$ is a subset of V such that for every edge $(v_1, v_2) \in E$, if $v_1 \notin T$ then $v_2 \in T$ and vice-versa. A minimal vertex cover is a vertex cover which is minimal with respect to inclusion.

2. The set of zero-divisors for M is the union of all associated primes of M.

Proof. See [18, p. 38 Theorem 6.1]. $\square$

Proposition 1.2.2. Let R be a Noetherian ring and M a finite R-module. Then

1. M has finitely many associated primes
2. $\mathrm{Ass}(M) \subseteq \mathrm{Supp}(M)$
3. Minimal elements of $\mathrm{Ass}(M)$ are exactly the minimal primes of $\mathrm{Supp}(M)$
4. $\sqrt{0 : M} = \bigcap_{\mathfrak{q} \in \min \mathrm{Ass}(M)} \mathfrak{q} = \bigcap_{\mathfrak{p} \in \min \mathrm{Supp}(M)} \mathfrak{p}$

Proof. See [18, p. 39 Theorem 6.5]. $\square$

Corollary 1.2.3. Let R be a Noetherian ring and I an $\mathfrak{m}$-primary ideal where $\mathfrak{m}$ is a maximal ideal of R. Then the quotient R/I is an Artinian ring.

Proof. By definition we know $\mathfrak{m} = \sqrt{I} = \sqrt{0 \colon R/I}$, it follows from proposition 1.2.2 4. that the only prime ideal of R containing I is the maximal ideal $\mathfrak{m}$. Since prime ideals of R that contain I are in bijection with prime ideals of R/I, we conclude R/I is a 0 dimensional Noetherian ring so the result follows by proposition 1.1.4. $\square$

Theorem 1.2.4. Let R be a Noetherian ring and I an ideal of R. Then for every $\mathfrak{p} \in \mathrm{Ass}(R/I)$ there exists a $\mathfrak{p}$-primary ideal $q_{\mathfrak{p}}$ such that

$$I = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(R/I)} q_{\mathfrak{p}}.$$

Moreover, $IR_{\mathfrak{p}} = q_{\mathfrak{p}} R_{\mathfrak{p}}$ for every minimal associated prime $\mathfrak{p}$.

Proof. See [1, p. 83 Theorem 7.13], [1, p. 52 Theorem 4.5] and [1, p. 54 Proposition 4.9]. $\square$

1.3 Graded rings and modules

We will now define the main algebraic objects of this text.

Definition 1.3.1. Let G be a monoid. A G-graded ring R is a ring such that

$$R = \bigoplus_{g \in G} R_g$$

where R_g are additive subgroups of R and $R_{g_1} R_{g_2} \subset R_{g_1 g_2}$. Note that if 0 is the neutral element of G, then R_0 has a ring structure since $R_0 R_0 \subset R_0$. Moreover, R_g is a R_0-module for every $g \in G$.

Definition 1.3.2. Let R be a G-graded ring. An R-module M is said to be a graded R-module if there are subgroups M_g of M such that

$$M = \bigoplus_{g \in G} M_g$$

and for every $g, h \in G$, $R_g M_h \subset M_{gh}$. Note that M_g is a R_0-module for every $g \in G$.

A graded submodule N of M is a submodule N such that $N = \bigoplus_{g \in G} N \cap M_g$.

Lastly, a R-algebra is said to be G-graded if in addition to being a graded R-module it is also a G-graded ring.

Definition 1.3.3. Let R be a G-graded ring and M a graded R-module. An element $f \in M$ is said to be homogeneous of degree g if $f \in M_g$.

Our goal in this section is to define the objects and maps for a category $\mathcal{M}_0^G(R)$ of graded R-modules, where R is a G-graded ring.

We already defined the objects of $\mathcal{M}_0^G(R)$. We will now define maps of this category.

Definition 1.3.4. Let R be a G-graded ring, M, N graded R-modules. A morphism $\varphi\colon M \to N$ in $\mathcal{M}_0^G(R)$ is a R-module homomorphism such that $\varphi(M_g) \subset N_g$ for every $g \in G$.

Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. The quotient M/N is still an object in $\mathcal{M}_0^G(R)$, that is, it is still a graded R-module. Given a morphism φ of $\mathcal{M}_0^G(R)$, then Ker φ and Im φ are graded R-modules.

The rings we will need in this text are either $\mathbb{N}^r$-graded rings or $\mathbb{Z}^r$-graded rings for some $r > 0$.

Proposition 1.3.1. Let R be a $\mathbb{Z}^r$-graded ring. Then R is a graded R_0-algebra and R is Noetherian if and only if R_0 is Noetherian and R is a finitely generated R_0-algebra.

Proof. The same proof as in [3, p.29 Proposition 1.5.5] works even if $r > 1$. □

We are specially interested in the case where R is a $\mathbb{N}^r$-graded algebra generated by finitely many elements of degree $e_1, \ldots, e_r$ where e_i represents the canonical generators of $\mathbb{N}^r$. Such algebras are called *standard*.

Another special case of interest comes from *shifting degrees*.

Definition 1.3.5. Let R be a $\mathbb{Z}^r$-graded ring and M a graded R-module. By definition, there are additive subgroups M_u of M such that

$$M = \bigoplus_{u \in \mathbb{Z}^r} M_u$$

Let $v \in \mathbb{Z}^r$, we denote by $M(v)$ the graded R-module such that $M(v)_u = M_{v+u}$ for every $u \in \mathbb{Z}^r$. We say $M(v)$ is the *shift* of M by v.

Definition 1.3.6. Let R be a ring, I an ideal of R and t a variable over R. The *Rees algebra of I* is the subring of $R[t]$ defined as

$$R[It] = \left\{ \sum_{i=0}^{n} a_i t^i \mid n \in \mathbb{N}, a_i \in I^i \right\} = \bigoplus_{n \geq 0} I^n t^n$$

where $I^0 = R$. Note that $R[It]$ is an $\mathbb{N}$-graded ring.

Definition 1.3.7. Let R be a ring, I an ideal of R and M a finite R-module. The module

$$\mathrm{gr}_I(M) = \bigoplus_{n \geq 0} \frac{I^n M}{I^{n+1} M}$$

is called the *associated graded module of M with respect to I*. Note that $\mathrm{gr}_I(R)$ is an $\mathbb{N}$-graded ring, we call it the *associated graded ring of R with respect to I*. It is also clear that $\mathrm{gr}_I(M)$ is a graded $\mathrm{gr}_I(R)$-module for any finite R-module M. Clearly we have $\dfrac{R[It]}{IR[It]} \cong \mathrm{gr}_I(R)$.

Definition 1.3.8. Let $(R, \mathfrak{m})$ be a local Noetherian ring with maximal ideal $\mathfrak{m}$. The *fiber cone of I* is the ring

$$\mathcal{F}_I(R) = \frac{R[It]}{\mathfrak{m}R[It]} \cong \frac{R}{\mathfrak{m}} \oplus \frac{I}{\mathfrak{m}I} \oplus \frac{I^2}{\mathfrak{m}I^2} \oplus \dots.$$

Clearly $\mathcal{F}_I(R) \cong \dfrac{\mathrm{gr}_I(R)}{\mathfrak{m}\mathrm{gr}_I(R)}$.

Moreover, the Krull dimension of $\mathcal{F}_I(R)$ is called the analytic spread of I and is denoted $s(I)$.

The Rees algebra will play an important role on the next sections. Given a ring R and an ideal I of R, one may ask how the dimension of R is related to the dimension of the Rees algebra of I. The answer to this question is the theorem below.

Theorem 1.3.2. Let R be a Noetherian ring and I an ideal of R. Then $\dim R$ is finite if and only if the dimension of the Rees algebra of I is finite. Moreover, if $\dim R$ is finite then:

$$\dim\, R[It] = \begin{cases} \dim\, R+1, & \text{if } I \not\subset P \text{ a prime ideal such that } \dim\,(R/P) = \dim\, R, \\ \dim\, R, & \text{otherwise.} \end{cases}$$

Proof. See [14, p. 99 Theorem 5.1.4 (1)]. □

1.4 Hilbert polynomials

Let R be a standard $\mathbb{N}$-graded algebra over an Artinian ring R_0 and M a finite graded R-module. We know from corollary 1.1.5 that for every $n \in \mathbb{N}$, M_n is a R_0-module of finite length. We are interested in the behaviour of $\ell(M_n)$ as n changes.

Definition 1.4.1. Let R be a standard $\mathbb{N}$-graded algebra over an Artinian ring R_0 and M a finite graded R-module. We call the numerical function $H(M, \_): \mathbb{N} \to \mathbb{N}$ such that $H(M,n) = \ell(M_n)$ for $n \in \mathbb{N}$ the *Hilbert function of M*.

In combinatorics, it is standard to work with the series whose coefficients are the values of a numerical function. The series $H_M(t) = \sum_{n=0}^{\infty} H(M,n)t^n$ is called the *Hilbert series of M*.

There are a few natural questions we may ask about the behaviour of Hilbert functions:

- Given a finite graded R-module M, is it possible to determine the *growth* of $H(M,n)$ based on information about M?
- Is there an upper bound to how fast the hilbert function of an arbitrary finite graded module M can grow?

The questions above are fully answered by the theorem below.

Theorem 1.4.1. Let R be a standard $\mathbb{N}$-graded algebra over an Artinian ring R_0 and M a finite graded R-module of Krull dimension d. Then there exists a polynomial $P_M(t) \in \mathbb{Q}[t]$ of degree $d-1$ such that $P_M(n) = H(M,n)$ for $n \gg 0$. The polynomial $P_M(t)$ is called the *Hilbert polynomial of* M.

Proof. See [3, p. 148 Theorem 4.1.3]. $\square$

The proposition below gives more information on the coefficients of $P_M(t)$.

Proposition 1.4.2. Let R be a standard $\mathbb{N}$-graded algebra over an Artinian ring R_0 and M a finite graded R-module of dimension d. Then there exists integers $a_0, \ldots, a_{d-1}$ such that

$$P_M(X) = \sum_{i=0}^{d-1} a_i \binom{X+i}{i}$$

Proof. See [3, p. 149 Lemma 4.1.4]. $\square$

Following the above notation, the *multiplicity of* M is defined to be

$$e(M) = \begin{cases} a_{d-1}, & \text{if } d > 0, \\ \ell(M), & \text{if } d = 0. \end{cases}$$

Example 1.4.2. Let $R = k[x_0, \ldots, x_n]$ where k is an arbitrary field. Then $P_R(t) = \binom{t+n}{n}$ for every $t > 0$. This follows from a standard *stars and bars* argument.

It is clear from proposition 1.1.6 that Hilbert function and polynomials are also additive on exact sequences of graded modules. Let $R = k[x_0, \ldots, x_n]$ and I a homogeneous ideal. From the exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0,$$

it is clear that $H(R/I,n) = H(R,n) - H(I,n)$ for every n. Assume now that I is a monomial ideal. From a combinatorial perspective, the problem of computing

$H(R/I, n)$ is equivalent to a *stars and bars* problem with *restrictions* given by the generators of I. A more concrete example is given below.

Example 1.4.3. Let $R = k[x_0, \ldots, x_n]$ and $I = (x_1 x_2)$. For every m, the number $H(R/I, m)$ can be interpreted as the number of ways m balls can be put into $n+1$ labeled boxes such that boxes 1 and 2 are not allowed to have balls inside them simultaneously.

Remark 1.4.1. For every graded ideal I in a polynomial ring over a field, using techniques from *Gröebner bases* it is possible to find a monomial ideal J such that I and J have the same Hilbert function. For more details see [11, p. 92 Corollary 6.1.5].

1.5 Hilbert-Samuel polynomials

Last section our setting was: R a standard $\mathbb{N}$-graded algebra over an Artinian ring R_0 and we studied the behaviour of Hilbert functions of finite graded R-modules. Our goal in this section is to study more general numerical functions called *Hilbert-Samuel functions*. Note that the setting will be different from the last one.

Before defining the main numerical function of this section we need the lemma below.

Lemma 1.5.1. Let $(R, \mathfrak{m})$ be a local Noetherian ring, I a $\mathfrak{m}$-primary ideal and M a finite R-module. Then $\ell(M/I^{n+1}M) = \sum_{i=0}^{n} \ell(I^i M/I^{i+1}M)$.

Proof. Consider the exact sequences of R-modules:

$$0 \longrightarrow \frac{M}{IM} \longrightarrow \frac{M}{IM} \longrightarrow \frac{M}{M} \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow \frac{I^{n-1}M}{I^nM} \longrightarrow \frac{M}{I^nM} \longrightarrow \frac{M}{I^{n-1}M} \longrightarrow 0$$

$$0 \longrightarrow \frac{I^nM}{I^{n+1}M} \longrightarrow \frac{M}{I^{n+1}M} \longrightarrow \frac{M}{I^nM} \longrightarrow 0$$

Since I is $\mathfrak{m}$-primary, we know $\mathfrak{m}^v \subseteq I$ for some $v > 0$. This implies that every module above is annihilated by a power of $\mathfrak{m}$ and thus are Artinian. The modules are Artinian and finitely generated, so we know every module above has finite length. By proposition 1.1.6, for every i (in particular for $i = 0$) we have

$$\ell(M/I^{i+1}M) = \ell(M/I^iM) + \ell(I^iM/I^{i+1}M).$$

If we assume the result is true for $k = n - 1$ then clearly

$$\begin{aligned}\ell(M/I^{n+1}M) &= \ell(M/I^nM) + \ell(I^nM/I^{n+1}M)\\ &= \sum_{j=0}^{n-1} \ell(I^jM/I^{j+1}M) + \ell(I^nM/I^{n+1}M)\end{aligned}$$

so the result is true for $k = n$ and the result follows by induction. $\square$

Definition 1.5.1. Let $(R, \mathfrak{m})$ be a local Noetherian ring, I an $\mathfrak{m}$-primary ideal and M a finite R-module. Let χ_M^I denote the numerical function such that $\chi_M^I(n) = \ell(M/I^{n+1}M)$. We call χ_M^I the *Hilbert-Samuel function of M with respect to I*.

Note that by lemma 1.5.1, $\chi_M^I(n) = \sum_{i=0}^n \ell(I^iM/I^{i+1}M) = \sum_{i=0}^n H(\mathrm{gr}_I(M), i)$[1]. One natural question is if there exists a polynomial $Q(t)$ in $\mathbb{Q}[t]$ such that $\chi_M^I(n) =$

[1]Because of this equality, the Hilbert-Samuel function is also called the *first iterated* Hilbert function.

$Q(n)$ for $n \gg 0$. As we will see in the theorem below, the answer to the question is positive.

Theorem 1.5.2. Let $(R, \mathfrak{m})$ be a Noetherian local ring, $M \neq 0$ a finite R-module of dimension d, and I an $\mathfrak{m}$-primary ideal. Then there exists a polynomial $Q(t) \in \mathbb{Q}[t]$ such that $Q(n) = \chi_M^I(n)$ for $n \gg 0$.

Proof. See [3, p.188 Proposition 4.6.2]. $\square$

Definition 1.5.2. The number $d! \lim_{n\to\infty} \dfrac{\ell(M/I^{n+1}M)}{n^d}$ is called $e(I, M)$, the *Hilbert-Samuel multiplicity of M with respect to I.*

Following the above notation, it is clear from lemma 1.5.1 and theorem 1.5.2 that there also exists a polynomial $P(n) = Q(n+1) - Q(n) = \ell(I^nM/I^{n+1}M)$ for $n \gg 0$. The lemma below gives us a formula for the Hilbert-Samuel multiplicity in terms of P.

Proposition 1.5.3. Let $P(n) \in \mathbb{Q}[n]$ of degree $d \geq 0$ with coefficients in $\mathbb{Z}$. Set $Q(n) = \sum_{i=0}^{n} P(i)$. Then $Q(n)$ is a polynomial in n of degree equal to $d+1$ and with coefficients in $\mathbb{Z}$. Moreover, if the leading coefficient of P is c, then the leading coefficient of Q is $c/(\deg P + 1)$.

In particular, If $(R, \mathfrak{m})$ is a local Noetherian ring, I a $\mathfrak{m}$-primary ideal and M a d dimensional finite R-module, then

$$e(I, M) = (d-1)! \lim_{n\to\infty} \frac{\ell(I^nM/I^{n+1}M)}{n^{d-1}}$$

Proof. See [14, p. 222 Lemma 11.1.2] for the first part. The last part follows directly by definition and from the first part. $\square$

1.6 Discrete geometry

In this section we introduce the combinatorial objects that will play an important role in the connection between multiplicities and volumes which is the topic of the next chapter.

Throughout this section, every vector space is a finite dimensional $\mathbb{R}$-vector space.

Definition 1.6.1. Let V be a vector space. An affine subspace (of dimension d) of V is a subset $u+W$ where W is a subspace (of dimension d) of V and $u \in V \backslash W$. The empty set is an affine subspace of dimension -1.

Moreover, if X is a subset of V, the *affine hull of* X, $\operatorname{aff}(X)$ is the smallest affine subspace of V containing X.

Definition 1.6.2. Let $X = \{x_1, \dots, x_n\}$ be a finite subset of a vector space V. We define the *convex hull of* X, $\operatorname{conv}(X)$ as

$$\operatorname{conv}(X) = \left\{ \sum_{i=1}^{n} a_i x_i \colon 0 \leq a_i \leq 1,\ i = 1, \dots, n,\ \sum_{i=1}^{n} a_i = 1 \right\}.$$

To define the objects that will be used later on, we need the following definition:

Definition 1.6.3. Let V and W be vector spaces. A map $f \colon V \to W$ is *affine* if $f(v) = g(v) + w_0$ for some linear map $g \colon V \to W$, for every $v \in V$ and $w_0 \in W$. Note that f is continuous with respect to the usual topology of $\mathbb{R}^n$.

If $W = \mathbb{R}$ we say f is an affine form.

Given an affine form α we call the preimage of $(0, \infty)$ (resp. $[0, \infty)$) the *open* (resp. *closed*) *halfspace defined by* α:

$$H_\alpha^{>} \colon = \{x \in V \colon \alpha(x) > 0\}, \quad H_\alpha^{<} = H_{-\alpha}^{>}$$

$$H_\alpha^+ : = \{x \in V : \alpha(x) \geq 0\}, \quad H_\alpha^- = H_{-\alpha}^+$$

Definition 1.6.4. If a subset P of V is the intersection of finitely many closed halfspaces, we say P is a polyhedron. If P is a bounded polyhedron, we say P is a polytope. The *dimension* of P is the dimension of $\mathrm{aff}(P)$.

Definition 1.6.5. Let P be a polyhedron. A hyperplane H is a *support hyperplane of* P if P is contained in one of the closed halfspaces bounded by H and $H \cap P \neq \emptyset$. The intersection $F = H \cap P$ is called a *face* of P.

Faces of polyhedra are also polyhedra. A face of dimension $\dim P - 1$ is called a *facet*, a face of dimension 0 is called a vertex and a face of dimension 1 is called an *edge*.

Polytopes in two dimensions are called *polygons*. The gray area in (a) is an example of a polytope P. The vertices of P are $(1,1), (1,6), (4,1)$, and $(3,5)$. The polytopes above are just the convex hull of their vertices, this is in fact always true:

Theorem 1.6.1. Let $P \subset V$. The following are equivalent:

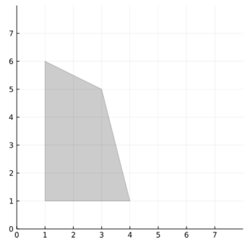
1. P is a polytope;

2. P is a polyhedron and P is the convex hull of its vertices;

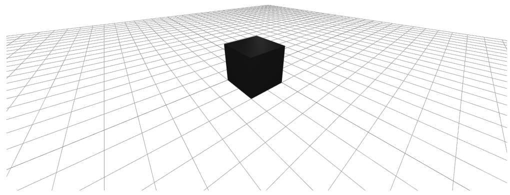
3. P is the convex hull of a finite subset of V.

Proof. See [7, p. 18 Theorem 1.26]. $\square$

There is also a theorem characterizing polyhedra that are not polytopes, but to state it we need two more definitions.



(a) A 2-dimensional polytope



(b) A 3-dimensional polytope

Definition 1.6.6. Let C be a subset of a vector space V. We say C is a conical set if it is closed under nonnegative linear combinations:

If $x_1, \ldots, x_n \in C$ and $a_1, \ldots, a_n \in \mathbb{R}$ such that $a_i \geq 0$, then $a_1x_1 + \ldots a_nx_n \in C$.

Suppose C is also finitely generated, that is, every element of C can be written as a nonnegative linear combination of $x_1, \ldots, x_n$. In this case[1], we say C is a *cone*.

Definition 1.6.7. Let V be a vector space and $A, B \subset V$. The *Minkowski sum* of A and B denoted by $A + B$ is:

$$A + B = \{a + b \,:\, a \in A,\ b \in B\}.$$

Moreover, we denote by nA the Minkowski sum of A with itself n times.

The proposition below will be useful later on:

Proposition 1.6.2. Let A and B be subsets of a vector space V. Then

$$\mathrm{conv}(A + B) = \mathrm{conv}(A) + \mathrm{conv}(B).$$

We can now characterize polyhedra based on polytopes and cones.

[1] It is also possible to give a definition of a cone based on linear halfspaces, for more details see [7, p. 11, Definition 1.14 and Theorem 1.15]

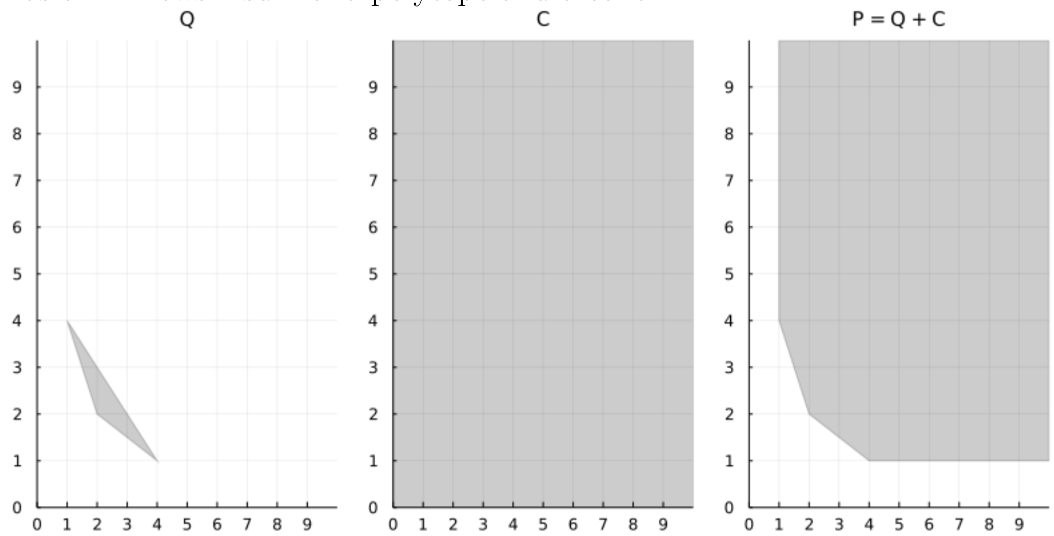
Theorem 1.6.3. Let P be a subset of a vector space V. The following are equivalent:

1. P is a polyhedron;

2. There exists a polytope Q and a cone C such that $P = Q + C$.

Proof. See [7, p. 19 Theorem 1.27]. $\square$

Example 1.6.8. Below is an example of one way to decompose a specific polyhedron as a Minkowski sum of a polytope and a cone:



As we will see later on, the theorem above plays an important role in relating different concepts of commutative algebra and algebraic geometry.

Lastly, we define the *Newton polytope* of a polynomial:

Definition 1.6.9. Let $f = \sum_{i=1}^{m} x^{\alpha_i} \in \mathbb{C}[x_1, \ldots, x_n]$. The *Newton polytope* of f denoted by $NP(f)$ is $\mathrm{conv}(\alpha_1, \ldots, \alpha_m) \subset \mathbb{R}^n$.

1.7 Monomial subrings

Now that we have defined polytopes, we are ready to define their algebraic counterparts. Let k be an arbitrary field, $R = k[x_1, \dots, x_n]$ the polynomial ring in n variables and $x^{\alpha_1}, \dots, x^{\alpha_n}$ monomials in R. The subring $k[x^{\alpha_1}, \dots, x^{\alpha_n}]$ of R is called a *monomial subring of R.*

One natural question that comes up is the relation between the Rees algebra of the ideal $(x^{\alpha_1}, \dots, x^{\alpha_n})$ and the monomial subring $k[x^{\alpha_1}, \dots, x^{\alpha_n}]$. Although it is clear that they are not the same ring, the monomial subring is, in a way, the fiber cone of the ideal $(x^{\alpha_1}, \dots, x^{\alpha_n})$.

Proposition 1.7.1. Let $I = (x^{\alpha_1}, \dots, x^{\alpha_s})$ be a monomial ideal of $R = k[x_1, \dots, x_n]$. Then

$$\mathcal{F}_I := \bigoplus_{n \in \mathbb{N}} \frac{I^n}{\mathfrak{m} I^n} \cong k[x^{\alpha_1}, \dots, x^{\alpha_s}]$$

where $\mathfrak{m} = (x_1, \dots, x_n)$.

Proof. The isomorphism follows from the natural map sending the residue class of $x^\alpha \in I^n$ in $I^n/\mathfrak{m}I^n$ to $x^\alpha \in k[x^{\alpha_1}, \dots, x^{\alpha_s}]$. $\square$

Let M be a submonoid of $\mathbb{N}^n$ and $k[M]$ the k-algebra generated by x^m where m ranges over the generators of M. For $a \in k$, $x^{m_1}, x^{m_2} \in k[M]$, the product in $k[M]$ is given by $ax^{m_1}x^{m_2} = ax^{m_1+m_2}$. The k-algebra $k[M]$ is called an *affine semigroup ring.*

Let M be a submonoid of $\mathbb{Z}^n$. Since for $x, y, z \in M$, $x + y = x + z$ implies $y = z$, it is possible to embed M in a group $\mathbb{Z}M$. The group $\mathbb{Z}M$ satisfies the following universal property:

There exists a homomorphism $i\colon M \to \mathbb{Z}M$ such that every homomorphism $\varphi\colon M \to H$ to a group H factors through $\mathbb{Z}M$ uniquely as $\varphi = \psi \circ i$ where

$\psi\colon \mathbb{Z}M \to H$ is a group homomorphism. Moreover, $\mathbb{Z}M$ is unique up to isomorphism.

The construction of $\mathbb{Z}M$ is, in many ways, very similar to localization in ring theory:

The group $\mathbb{Z}M$ is the set of equivalence classes: $\dfrac{M \times M}{\sim}$ where $(x, y) \sim (u, v)$ if and only if $x + v + z = u + y + z$ for some $z \in M$. The class (x, y) will be denoted by $x - y$. Addition in $\mathbb{Z}M$ is given by $(x-y)+(u-v) = (x+u)-(y+v)$. The map $i\colon M \to \mathbb{Z}M$ such that $i(x) = x - 0$ is a homomorphism of monoids that satisfies the universal property. Moreover, i is injective [1].

Note that by definition if $N \subset M$ are monoids, then $\mathbb{Z}N \subset \mathbb{Z}M$. It is also clear that if M is a group, $\mathbb{Z}M = M$.

From the remarks above, it is clear that for every submonoid M of $\mathbb{Z}^n$, the group $\mathbb{Z}M$ is a torsionfree finitely generated abelian group, and thus, $\mathbb{Z}M \cong \mathbb{Z}^r$ for some $0 \le r \le n$. The number r such that $\mathbb{Z}M \cong \mathbb{Z}^r$ is called the *rank of M* and will be denoted by $\operatorname{rank}(M)$ or $\operatorname{rank} M$.

To prove our first result on affine semigroup rings we need the following theorem:

Theorem 1.7.2. Let k be a field, $R = k[\alpha_1, \ldots, \alpha_n]$, $Q(R)$ the field of fractions of R and $r = \operatorname{trdeg}_k R$ the transcendence degree of the extension $k \subset Q(R)$.

Then $r = \dim R$.

Proof. See [18, p. 34 Theorem 5.6]. □

Proposition 1.7.3. Let M be a submonoid of $\mathbb{Z}^n$ and $k[M]$ its affine semigroup ring. Then

$$\dim k[M] = \operatorname{rank}(M).$$

[1]Injectivity follows from the property: $x, y, z \in M$, $x + y = x + z$ implies $y = z$.

Proof. By theorem 1.7.2 we know $\dim\, k[M] = \operatorname{trdeg}_k(k[M])$, assume that $\operatorname{rank}(M) = r$. It is clear that $Q(k[\mathbb{Z}M]) = Q(k[M])$. We already know that $\mathbb{Z}M = \mathbb{Z}^r$ for some r, so we conclude $Q(k[M]) \cong Q(k[\mathbb{Z}^r]) \cong Q(k[t_1, \ldots, t_r])$. From these isomorphisms we conclude $\dim\, k[M] = r$. □

Chapter 2

Multiplicities, volumes and positivity

In this chapter we are going to define a more general notion of *multiplicity*, which comes from a multidimensional version of Hilbert functions. We then explore the connection of these functions with their one dimensional counterparts, and introduce the concepts of reduction and integral closures of ideals. As will become clear, these notions are closely related to the theory of Hilbert functions.

Throughout this chapter, given a polytope P, the n-dimensional euclidean volume of P (that is, the integral of 1 in the region $P \subset \mathbb{R}^n$) will be denoted by $V_n(P)$.

2.1 The one dimensional case: Multiplicity and volume

The goal of this section is to introduce the combinatorial counterparts of Hilbert polynomials and give a purely algebraic interpretation of the notion of *volume* for polytopes. On section section 1.6 we defined polytopes, which turn out to be the convex hull of a finite set of points in $\mathbb{R}^n$ for some $n > 0$. Polytopes that are of interest to us are called *rational polytopes*, that is, polytopes such that its vertices have only rational coordinates. Since the coordinates of each vertex of a rational polytope P are rational, there exists n such that the vertices of nP have integer coordinates, and thus from now on every vertex of every polytope will be assumed to have integer coordinates.

Let P be a polytope and $\mathcal{E}(P) = \#\big(P \cap \mathbb{Z}^m\big)$ the number of lattice points inside P. Since $\mathcal{E}(P) = \mathcal{E}(Q)$ if Q is a translation of P, we always assume the vertices of P have nonnegative coordinates. Our goal in this section is to understand how does $\mathcal{E}(nP)$ change when n changes. For this we need the following definition:

Definition 2.1.1. Let $(x_1, \ldots, x_d) \in \mathbb{Z}^m \subset \mathbb{R}^m$ and $P = \operatorname{conv}(x_1, \ldots, x_d)$. We call the numerical function $E(P, \_)\colon \mathbb{N} \to \mathbb{N}$ such that $E(P, n) = \mathcal{E}(nP)$ the *Ehrhart function of* P. Note that $\#\big(P \cap \frac{1}{n}\mathbb{Z}^m\big) = \#\big(nP \cap \mathbb{Z}^m\big)$, hence $E(P, n) = \#\big(P \cap \frac{1}{n}\mathbb{Z}^m\big)$.

As mentioned before, we also define the series $\sum_{n=1}^{\infty} E(P, n)t^n$ the *Ehrhart series of* P.

Let P be an m-dimensional polytope in $\mathbb{R}^m$ and let $E(P)_k = P \cap \frac{1}{k}\mathbb{Z}^m$. Set $S_k = \sum_{e \in E(P)_k} k^{-m}$, that is, S_k is the sum of volumes of hypercubes centered on points of $E(P)_k$. As k grows larger, it is clear that S_k converges to $V_m(P)$, in other words:

$$V_m(P) = \lim_{n\to\infty} \frac{E(P, n)}{n^m}.$$

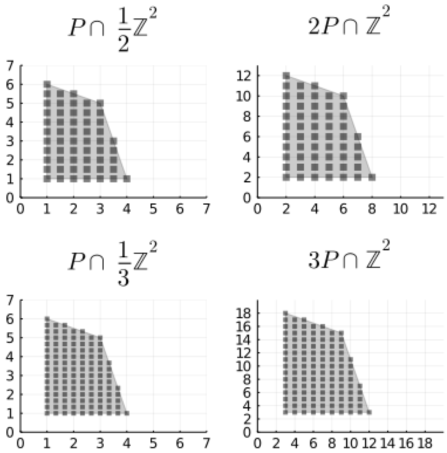


Figure 2.1: A visualization of the bijection between $P \cap \dfrac{1}{n}\mathbb{Z}^m = nP \cap \mathbb{Z}^m$ for the polytope with vertices: $(1,1),(1,6),(4,1),(3,5)$ and $n = 2,3$.

Definition 2.1.2. Let $P \subset \mathbb{R}^d$ be a polytope such that the vertices of P are $x_1,\dots,x_n$. Consider the cone $C(P) \subset \mathbb{R}^{n+1}$ generated by $(x_1,1),\dots,(x_n,1)$. The cone $C(P)$ is called *the cone over* P. It is clear that $C(P)$ is a monoid. We will denote by $k[P]$ the monomial algebra $k[C(P)\cap\mathbb{N}^{n+1}]$, $k[P]$ is called the monomial algebra of the cone over P.

Given a polytope $P \subset \mathbb{R}^n$, denote by $\alpha_1,\dots,\alpha_r \in \mathbb{Z}^n$ the lattice points of P. Since $k[P]$ is generated by $x^{(\alpha_1,1)},\dots,x^{(\alpha_r,1)}$, we can give the following grading to $k[P]$ to make it a standard $\mathbb{N}$-graded k-algebra:

$$\deg\, x^{\alpha} = \deg\, x_1^{\alpha_1}\dots x_n^{\alpha_n}x_{n+1}^{\alpha_{n+1}} = \alpha_{n+1}.$$

Let $P \subset \mathbb{R}^n$ be a polytope. By definition, $k[P]$ is a standard $\mathbb{N}$-graded algebra over an Artinian ring (more specifically, a field). The set $\{x^{(a,i)} \colon a \in iP\cap\mathbb{Z}^n\}$ is a k-basis for $k[P]_i$, so we conclude the dimension of $k[P]_i$ as a k-vector space is exactly $E(P,i)$. This equality implies $H(k[P],n) = E(P,n)$. From these equalities

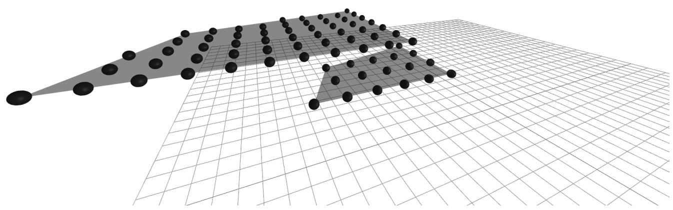


Figure 2.2: The spheres illustrate a k-basis for $k[P]_1 \oplus k[P]_2$ for a given polygon P.

we can compute the euclidean volume of arbitrary polytopes using only algebraic objects:

Theorem 2.1.1. Let $P \subset \mathbb{R}^m$ be an m-dimensional polytope. Then

$$V_m(P) = \frac{e(k[P])}{m!}.$$

Proof. From previous results in this section we have the following equalities:

1. $V_m(P) = \lim_{i\to\infty} \dfrac{E(P,i)}{i^m}$.
2. $E(P,i) = H(k[P],i)$.

From section section 1.4 we know $0 < \lim_{i\to\infty} \dfrac{H(k[P],i)}{i^m} < \infty$ implies $m = \dim k[P] - 1$ and in particular,

$$\lim_{i\to\infty} \frac{H(k[P],i)}{i^m} = \frac{e(k[P])}{m!}.$$

□

Example 2.1.3. Consider the polytope P from fig. 1.1a. Using the computer algebra system `Macaulay2`, we can compute the Hilbert polynomial of the monomial algebra $k[P]$ and so we conclude the volume of P is 11.

We can also use the package `Polyhedra.jl` from the `Julia` programming language and compute the multiplicity of $k[P]$ from the volume of P.

Remark 2.1.1. In this section we defined the *Ehrhart function* of polytopes. Given a polytope P we can also define a similar function denoted by $E^+(P, \_)$ which counts only lattice points that lie *in the interior of* P. This function can be fully determined by $E(P, \_)$ and it is possible to define a $k[P]$-module ω to give an algebraic interpretation of the connection between $E(P, \_)$ and $E^+(P, \_)$. The module ω is called the *canonical module of* $k[P]$. For more details see [7, Chapter 6].

2.2 Hilbert functions and mixed multiplicities

In this section we generalize the concepts of sections 1.4 and 1.5 to $\mathbb{N}^2$-graded rings. The results from this section hold for two kinds of rings: local rings and standard $\mathbb{N}^2$-graded algebras over Artinian rings. Both settings are analogous to the ones in sections 1.4 and 1.5. If a property holds for every $(n, m) \in \mathbb{N}^2$ such that $n \geq a_1, m \geq a_2$, then we say the propert holds for $(n, m) \gg 0$.

Definition 2.2.1. Let $R = \bigoplus_{u \in \mathbb{N}^2} R_u$ a standard $\mathbb{N}^2$-graded algebra over an Artinian local ring R_0. The function $H_R(u) = \ell(R_u)$ is called the *Hilbert function of R.*

Following the definition above, the same result mentioned in section 1.4 holds:

Proposition 2.2.1. There exists a polynomial $P_R \in \mathbb{Q}[t_0, t_1]$ such that $H_R(u) = P_R(u)$ for $u \gg 0$. Moreover, if $P_R(u) \neq 0$, then we can write $P_R(u)$ as

$$P_R(u) = \sum_{\alpha \in \mathbb{N}^2, |\alpha| = r} \frac{1}{\alpha!} e_\alpha(R) u^\alpha + \text{terms of degree lower than } r.$$

The polynomial P_R is called the *Hilbert polynomial of* R. The coefficients $e_\alpha(R)$ are called the *mixed multiplicities of* R. In the case where R is a standard $\mathbb{N}$-graded algebra, then R has only one mixed multiplicity, and it is the multiplicity of R which we denote by $e(R)$.

We know from previous results that the multiplicity $e(R)$ is always positive, but this is not the case for mixed multiplicities, it is possible that for some α, $e_\alpha(R) = 0$.

Example 2.2.2. Let $R = k[x_1, x_2, y]$ be a standard $\mathbb{N}^2$-graded algebra over a field k, where deg $x_i = (1,0)$ and deg $y = (0,1)$. Given a vector $(u,v) \in \mathbb{N}^2$, the set $\{x_1^{u-i}x_2^i y^v \mid i = 0, \ldots, u\}$ is a basis for $R_{(u,v)}$ and so we conclude $\ell(R_{(u,v)}) = u+1 = P_R(u,v)$. In particular, the only nonzero mixed multiplicity of R is $e_{(1,0)}(R) = 1$.

Next we define a specific standard $\mathbb{N}$-graded subalgebra that is useful in studying certain mixed multiplicites:

Definition 2.2.3. Let R be a standard $\mathbb{N}^2$ graded algebra over an Artinian local ring R_0. Let $\lambda = (\lambda_0, \lambda_1) \in \mathbb{N}^2$. Set

$$R^\lambda = \bigoplus_{n \in \mathbb{N}} R_{n\lambda}.$$

Then R^λ is a standard $\mathbb{N}$-graded algebra. We call R^λ the λ-*diagonal subalgebra of* R.

Lemma 2.2.2. Following the above notation, let $r = \deg P_R(u,v) \geq 0$ and assume $\lambda_0, \lambda_1 > 0$. Then dim $R^\lambda = r+1$ and

$$e(R^\lambda) = r! \sum_{(n,m)\in\mathbb{N}^2, n+m=r} \frac{1}{n!m!} e_{(n,m)}(R)\lambda_0^n \lambda_1^m.$$

Proof. Since the coordinates of λ are positive, we have the following equalities:

$$P_{R^\lambda}(n) = P_R(n\lambda) = n^r \sum_{\alpha\in\mathbb{N}^{s+1}, |\alpha|=r} \frac{1}{\alpha!} e_\alpha(R)\lambda^\alpha + \text{terms of degree lower than } r.$$

In particular, by theorem 1.4.1, $\dim R^\lambda = r+1$ and the result follows from the definition of $e(R^\lambda)$. $\square$

We now generalize the notion of associated graded ring defined previously to standard $\mathbb{N}^2$-graded algebras.

Definition 2.2.4. Let $(A, \mathfrak{m})$ be either a local ring or a standard $\mathbb{N}$-graded algebra over a field, where $\mathfrak{m}$ is the maximal graded ideal. Let I be an $\mathfrak{m}$-primary ideal and J an arbitrary ideal of A. Set

$$R(I|J) = \bigoplus_{(u,v)\in\mathbb{N}^2} I^u J^v / I^{u+1} J^v.$$

Clearly $R(I|J)$ is a standard $\mathbb{N}^2$-graded algebra over an Artinian ring (the base ring $R_0 = A/I$ is Artinian by corollary 1.2.3). The mixed multiplicities of $R(I|J)$ will be denoted by $e_i(I|J) = e_{(n-i,i)}(I|J)$, where n is the degree of the Hilbert function of $R(I|J)$.

Following the above notation, the ring $R(I|J)$ can be thought of as the associated graded ring of the Rees algebra $A[Jt]$ with respect to the ideal I.

Example 2.2.5. Let $A = \mathbb{C}[x,y,z,w,t]$ and let J be the jacobian ideal of the polynomial $xyzwt$, that is, the ideal generated by the partial derivatives of $xyzwt$. Then $e_{(4,0)}(\mathfrak{m}|J) = 1, e_{(3,1)}(\mathfrak{m}|J) = 4, e_{(2,2)}(\mathfrak{m}|J) = 6, e_{(1,3)}(\mathfrak{m}|J) = 4, e_{(0,4)}(\mathfrak{m}|J) = 1$. Note that $e_{(4-i,i)}(\mathfrak{m}|J) = \binom{4}{i}$.

Example 2.2.6. Let $A = \mathbb{C}[x,y,z,w]$ and let J be the jacobian ideal of the polynomial $xyzw(x+y+z+w)$. Then $e_{(3,0)}(\mathfrak{m}|J) = 1, e_{(2,1)}(\mathfrak{m}|J) = 4, e_{(1,2)}(\mathfrak{m}|J) = 6, e_{(0,3)}(\mathfrak{m}|J) = 4$.

2.3 Positivity of mixed multiplicities

In this section we classify the positive multiplicities of the algebra $R(I|J)$. We will first give a condition to when a specific mixed multiplicity is positive, and then use the condition as a base case for every mixed multiplicity.

Definition 2.3.1. Let R be a Noetherian ring and I an ideal of R. The ideal

$$(0\colon I^{\infty}) = \{a \in R \mid aI^n = 0 \text{ for some n } \in \mathbb{N}\} = \bigcup_{n\in\mathbb{N}} (0\colon I^n)$$

is called the *saturation of* I.

We will need the following lemma:

Lemma 2.3.1. Let I be an ideal of a Noetherian ring R. Then $(0\colon I^{\infty}) = 0$ if and only if I contains a nonzero divisor. In particular, if $(0\colon I^{\infty}) = 0$, then I has positive height.

Proof. See [2, p. 17 Lemma 2.1.1] for the first part. The second part follows from proposition 1.2.1 and proposition 1.2.2 3. □

Let $(A, \mathfrak{m})$ be either a local Noetherian ring or a standard $\mathbb{N}^2$-graded algebra over a field, where $\mathfrak{m}$ is the maximal graded ideal. Moreover, let I be a $\mathfrak{m}$-primary ideal and J be an arbitrary ideal of A. Set $R = R(I|J)$. We fix this notation for the rest of this section.

The following theorem will play the role of base case later in this section:

Theorem 2.3.2. Assume that $d = \dim A/(0\colon J^{\infty}) \geq 1$. Then

1. $\deg P_R(u, v) = d - 1$,
2. $e_{(d-1,0)}(I|J) = e(I, A/(0\colon J^{\infty}))$.

Proof. Let I', J' be the ideals generated by I, J in the quotient ring $A/(0\colon J^\infty)$ and set $R' = R(I'|J')$.

Given M, N, M', Q A-modules such that $N \subset M' \subset M$, consider the following isomorphisms:

1. $\dfrac{\frac{M}{N}}{\frac{M'}{N}} \cong \dfrac{M}{M'}$

2. $\dfrac{Q + M + IQ}{M + IQ} \cong \dfrac{Q}{Q \cap M + IQ}$

From these isomorphisms we get:

$$
\begin{aligned}
R'_u &= (I^u J^v + (0\colon J^\infty))/(I^{u+1} J^v + (0\colon J^\infty)) \\
&= I^u J^v/(I^{u+1} J^v + I^u J^v \cap (0\colon J^\infty)).
\end{aligned}
$$

But since A is Noetherian, the increasing chain $(0\colon J) \subset (0\colon J^2) \subset \dots$ must satisfy $(0\colon J^i) = (0\colon J^j)$ for every $i, j \gg 0$.

Let $x \in I^u J^v \cap (0\colon J^\infty)$, $(u, v) \gg 0$ and $m \in \mathbb{N}$ such that $(0\colon J^i) = (0\colon J^j)$ for every $\forall i, j \geq m$. Since $xJ^t = 0$ for some $t \leq m$, rearranging the product we have $I^u J^{v-t} J^t = I^u J^v$ and $x \in I^u J^v \subset I^u J^{v-t}$, therefore

$$I^u J^v \cap (0\colon J^\infty) = 0.$$

From the equality above it is clear that $R'_{(u,v)} = R_{(u,v)}$ hence $P_R(u, v) = P_{R'}(u, v)$. This equality of Hilbert polynomials implies we can replace A by $A/(0\colon J^\infty)$. In replacing A by $A/(0\colon J^\infty)$ we are also reducing $(0\colon J^\infty)$ to 0 and $d = \dim A \geq 1$.

Since $(0\colon J^\infty) = 0$, the lemma above implies J has positive height. Let $\lambda = (1, 1)$ and consider the λ-diagonal subalgebra

$$R^\lambda = \bigoplus_{n\geq 0} \frac{I^n J^n}{I^{n+1}J^n} \cong \frac{A[IJt]}{(I)}.$$

Note that ht $IJ = \min\{\text{ht } I,\ \text{ht } J\} \geq 1$ (the inequality holds since I is $\mathfrak{m}$-primary and $\dim A \geq 1$). By theorem 1.3.2 we get $\dim A[IJt] = d+1$. Since R^λ is a quotient of $A[IJt]$ and I is not contained in any of the minimal primes of A, we conclude $\dim R^\lambda \leq d$. By lemma 2.2.2 we get deg $P_R(u,v) \leq d-1$.

It is clear that $\dim A/J^m +$ ht $J^m \leq \dim A$ and ht $J^m \geq 1$ for any $m \geq 1$, we conclude $\dim A/J^m < d$ and thus $e(I,A) = e(I,J^m)$.

Finally, for $m \gg 0$ we have the following equalities:

$$e(I,A) = e(I,J^m) = \lim_{n\to\infty} \frac{\ell(I^nJ^m/I^{n+1}J^m)}{n^{d-1}/(d-1)!} = \lim_{n\to\infty} \frac{P_R(n,m)}{n^{d-1}/(d-1)!}$$

where the second equality follows from proposition 1.5.3 and $(0 : J^\infty) = 0$ (so the dimension of J^m is d) and the last one by definition.

The Hilbert-Samuel multiplicity of A is nonzero and thus we get deg $P_R(u,v) = d-1$. computing the limit it is clear that $e(I,A) = e_{(d-1,0)}(I|J)$. □

To get a similar result for any multiplicity, we need two more definitions:

Definition 2.3.2. Let S be a standard $\mathbb{N}^2$-graded algebra. We say a sequence of elements $z_1,\ldots,z_n \in S$ is *filter-regular* if for $(u,v) \gg 0$ and any i:

$$[(z_1,\ldots,z_{i-1})\colon z_i]_{(u,v)} = (z_1,\ldots,z_{i-1})_{(u,v)}.$$

Filter-regular sequences are very similar to regular sequences in high degree. The next proposition gives us another possible definition of filter-regular sequences:

Proposition 2.3.3. Let S be a standard $\mathbb{N}^2$-graded algebra over an Artinian local ring S_0 and $z_1,\ldots,z_s$ homogeneous elements of S. The sequence $z_1,\ldots,z_s$ is filter-regular if and only if $z_i \notin \mathfrak{p}$ for all associated prime ideals $\mathfrak{p} \not\supset S_+$ of

$S/(z_1, \dots, z_{i-1})$, where S_+ is the ideal generated by elements of degree (u, v) such that $u, v \geq 1$.

Proof. See [23, Lemma 1.2]. $\square$

Definition 2.3.3. Set S as the $\mathbb{N}^2$-graded algebra:

$$S = \bigoplus_{(u,v)\in\mathbb{N}^2} I^u J^v / I^{u+1} J^{v+1}.$$

Let $x_1, \dots, x_m$ be a sequence of elements of J. Denote by x_i^* the residue class of x_i in J/IJ^2. We say $x_1, \dots, x_m$ is a superficial sequence for the ideal J (with respect to I) if $x_1^*, \dots, x_m^*$ is a filter-regular sequence in S.

At some point we will also call a sequence of elements $y_1, \dots, y_n$ of I superficial if the residue classes $y_1^*, \dots, y_n^*$ of the y_i in S is a filter-regular sequence in S.

Before proving the main theorem of this section we need the following lemma that will be useful for the inductive step:

Lemma 2.3.4. Let $Q = (x)$ be an ideal of A generated by a superficial sequence of J. Let $\bar{I}, \bar{J}$ be the ideals generated by I, J in the quotient ring A/Q and set $\bar{R} = R(\bar{I}|\bar{J})$. Then

$$P_{\bar{R}}(u, v) = P_R(u, v) - P_R(u, v-1).$$

Proof. We know by definition that $(0 : x^*)_{(u,v)} = 0$ for $(u, v) \gg 0$ where x^* is the residue class of x in J/IJ^2.

From the product rule in S we conclude

$$(I^{u+1}J^{v+2} : x) \cap I^u J^v = I^{u+1}J^{v+1} \quad (2.1)$$

$$(I^{u+1}J^{v+2} : x) \cap I^u J^{v+1} = I^{u+1}J^{v+1} \quad (2.2)$$

Note that the last equality also means $(0:\hat{x})_{(u,v+1)} = 0$ for every $(u,v) \gg 0$, where $\hat{x}$ is the residue class of x in $R(I|J)$. In particular, $P_{R/(0:\hat{x})}(u,v) = P_R(u,v)$. From the short exact sequence:

$$0 \longrightarrow \frac{R}{(0:\hat{x})}(0,-1) \xrightarrow{\hat{x}} R \longrightarrow \frac{R}{\hat{x}R} \longrightarrow 0$$

we conclude $P_{R/\hat{x}R}(u,v) = P_R(u,v) - P_R(u,v-1)$. Our goal now is to prove that $R/\hat{x}R_{(u,v)} = \bar{R}_{(u,v)}$ for $(u,v) \gg 0$.

By eq. (2.1) we have the following equality:

$$I^{u+1}J^{v+1} = \{y \in I^u J^v | xy \in I^{u+1}J^{v+2}\}$$

that is, $I^{u+1}J^{v+1}$ is exactly the subset of elements of xI^uJ^v that are in $I^{u+1}J^{v+2}$. This means

$$I^{u+1}J^{v+2} \cap xI^uJ^v = xI^{u+1}J^{v+1} \quad \text{for } (u,v) \gg 0.$$

By the Artin-Rees lemma (see [18, p. 63 Exercise 8.8]) there exists $u_0, v_0 \in \mathbb{N}$ such that

$$I^uJ^v \cap (x) \subseteq xI^{u-u_0}J^{v-v_0}$$

for $(u,v) \geq (u_0, v_0)$. Therefore,

$$\begin{aligned} I^uJ^v \cap (x) &= I^uJ^v \cap xI^{u-u_0}J^{v-v_0} \\ &= xI^uJ^{v-1} \end{aligned}$$

for $(u,v) \gg 0$.

From the isomorphisms mentioned in theorem 2.3.2 and the equality we just proved, for $(u,v) \gg 0$ we have

$$
\begin{aligned}
\bar{R}_{(u,v)} &= \frac{I^u J^v + (x)}{I^{u+1}J^v + (x)} \\
&= \frac{I^u J^v}{I^{u+1}J^v + (x) \cap I^u J^v} \\
&= \frac{I^u J^v}{I^{u+1}J^v + xI^u J^{v-1}} \\
&= (R/(\hat{x}))_{(u,v)}.
\end{aligned}
$$

In particular, $P_{\bar{R}}(u,v) = P_{R/\hat{x}R}(u,v) = P_R(u,v) - P_R(u,v-1)$. $\square$

Following the notation above, let $Q' = (x,y)$ be an ideal of A generated by a superficial sequence of J. Let $\bar{y}$ denote the class of y in A/Q.

Let $\bar{S} = \oplus_{(u,v)\in\mathbb{N}^2}(\bar{I})^u(\bar{J})^v/(\bar{I})^{u+1}(\bar{J})^{v+1}$. Then for $u, v \gg 0$:

$$
\begin{aligned}
[S/x^*]_{(u,v)} &= I^u J^v/(I^{u+1}J^{v+1} + xI^u J^{v-1}) \\
&= I^u J^v/(I^{u+1}J^{v+1} + (x) \cap I^u J^v) \\
&= (I^u J^v + (x))/(I^{u+1}J^{v+1} + (x)) \\
&= \bar{S}_{(u,v)}.
\end{aligned}
$$

Since $[(x^*) : y]_u = (x^*)$ (x, y is a superficial sequence), we also have $[0_{S^*} : y]_u = 0_{S^*}$. Therefore the ideal $\bar{Q}' = (\bar{y})$ of A/Q is generated by a superficial sequence for $\bar{J}$ (with respect to $\bar{I}$). In particular, we can apply the lemma above inductively for $Q = (x_1, \ldots, x_m)$.

The theorem below is a sufficient and necessary condition for positivity of mixed multiplicities.

Theorem 2.3.5. Let $Q = (x_1, \ldots, x_i)$ be any ideal generated by a superficial sequence of J. Then $e_i(I|J) > 0$ if and only if

$$\dim A/(Q : J^\infty) = \dim A/(0 \colon J^\infty) - i.$$

In this case,

$$e_i(I|J) = e(I, A/(Q : J^\infty)).$$

Proof. If $i = 0$, then $Q = 0$ and the result follows from theorem 2.3.2.

If $i > 0$, let $\bar{R}, \bar{I}, \bar{J}$ be as in lemma 2.3.4. Then $\deg P_{\bar{R}} \leq \dim A/(0 : J^\infty) - 1 - i = r$. Writing

$$P_{\bar{R}}(u, v) = \sum_{(\alpha,\beta)\in\mathbb{N}^2, \alpha+\beta=r} \frac{e_{(\alpha,\beta)}(\bar{I}|\bar{J})}{\alpha!\beta!} u^\alpha v^\beta + \ \{\text{lower degree terms}\}.$$

Then

$$e_{(r,i)}(I|J) = e_{(r,0)}(\bar{I}|\bar{J}).$$

From the equality above, $e_{(r,i)}(I|J) > 0$ if and only if $e_{(r,0)}(\bar{I}|\bar{J}) > 0$.

Therefore, $\deg P_{\bar{R}}(u, v) = r$ and by theorem 2.3.2, $\dim A/(Q\colon J^\infty) = r + 1$.

On the other hand, if $\dim A/(Q\colon J^\infty) = r + 1$, then

$$e_{(r,0)}(\bar{I}|\bar{J}) = e\Big(\bar{I}, \frac{A/Q}{(0\colon \bar{J}^\infty)}\Big) = e(I, A/(Q\colon J^\infty)).$$

Since the Hilbert-Samuel multiplicity is always positive we conclude $e_{(r,i)}(I|J) = e_{(r,0)}(\bar{I}|\bar{J}) > 0$. $\square$

Corollary 2.3.6. Let I be an $\mathfrak{m}$-primary ideal. Then $e_i(\mathfrak{m}|J) > 0$ if and only if $e_i(I|J) > 0$.

Proof. The dimension of the ring $A/(Q : J^\infty)$ does not depend on I. $\square$

The condition for positivity of mixed multiplicities requires an intricate definition of superficial sequences. The version of the result that will be useful later on is based on the next results:

Definition 2.3.4. Let k be the residue field of A. We say a property holds for *a general element x of an ideal* $Q = (x_1, \ldots, x_m)$ if there exists a nonempty Zariski-open subset $U \subset k^m$ such that whenever $x = \sum_{j=1}^{m} c_j x_j$ and the image of $(c_1, \ldots, c_m)$ in k^m belongs to U, then the property holds for x.

Lemma 2.3.7. Assume that k is infinite. Any sequence which consists of general elements in J is a superficial sequence for J.

Moreover, if $(S, \mathfrak{m})$ is a standard $\mathbb{N}^2$-graded algebra and f is a general linear form in S, then the residue class of f in $\bigoplus_{u,v \in \mathbb{N}} \mathfrak{m}^u J^v / \mathfrak{m}^{u+1} J^{v+1}$ is filter-regular.

Proof. See [24, Lemma 1.5]. □

Corollary 2.3.8. Assume that the local ring A has infinite residue field. Let Q be an ideal generated by i general elements in J. Then $e_i(I|J) > 0$ if and only if $\dim A/(Q : J^\infty) = \dim A/(0 : J^\infty) - i$. In this case,

$$e_i(I|J) = e(I, A/(Q : J^\infty)).$$

Proof. This is a direct consequence of theorem 2.3.5 and lemma 2.3.7. □

Remark 2.3.1. Although every result in this section is for the standard $\mathbb{N}^2$-graded algebra $R(I|J)$, in [23] the author proved similar results for standard $\mathbb{N}^2$-graded algebras over Artinian rings.

In [24] the authors proved the results in this section for a sequence of ideals $J_1, \ldots, J_s$ and the standard $\mathbb{N}^{s+1}$-graded algebra $R(I|J_1, \ldots, J_s)$.

Remark 2.3.2. In [24] the authors also prove that given a system of polynomials $F : f_1 = \cdots = f_n = 0$ in $\mathbb{C}[x_1, \ldots, x_n]$, the number of distinct solutions of F in $(\mathbb{C}^*)^n$ is bounded above by certain mixed multiplicities. They also prove the famous Bernstein theorem, which gives an upper bound to the number of solutions of F based on the Newton polytopes of the f_i.

2.4 Integral closure of (monomial) ideals

Let I be an $\mathfrak{m}$-primary ideal of a Noetherian local ring $(R, \mathfrak{m})$. One may ask what elements $r \in R$ satisfy the equality $e(I, R) = e(I+rR, R)$. This question is directly connected to the integral closure of ideals.

Definition 2.4.1. Let R be a ring and I an ideal of R. An element $r \in R$ is called *integral over* I if there exists $a_i \in I^i$, $i = 1, \ldots, n$ such that

$$r^n + \sum_{i=1}^{n} r^{n-i} a_i = 0.$$

The equality above is called an *integral dependence equation of* r *over* I.

Moreover, the integral closure $\bar{I}$ of I is the set of all elements of r that are integral over I.

Proposition 2.4.1. Let I be an ideal of a ring R. Then the integral closure of I is an ideal of R.

Proof. See [14, p. 6 Corollary 1.3.1]. □

Example 2.4.2. Let $R = k[x, y]$ and $I = (x^2, y^2)$. Then $\bar{I} = (x^2, y^2, xy)$.

The integral closure of monomial ideals is of particular interest.

Proposition 2.4.2. Let I be a monomial ideal in $k[x_1, \ldots, x_n]$. Then the integral closure of I is also a monomial ideal.

Proof. See [14, p. 9 Proposition 1.4.2]. □

Since the integral closure of a monomial ideal is monomial, it is natural to ask what combinatorial properties of I are preserved by taking the integral closure. We need the following definition:

Definition 2.4.3. Let $I = (x^{\alpha_1}, \ldots, x^{\alpha_s})$ be a monomial ideal of $k[x_1, \ldots, x_n]$. The exponent set of I is the set of all $v \in \mathbb{N}^n$ such that $x^v \in I$.

The *Newton polyhedron* of I denoted by $NP(I)$ is the convex hull of the exponent set of I in $\mathbb{R}^n$.

Note that the exponent set of $I = (x^{\alpha_1}, \ldots, x^{\alpha_s})$ is $\mathbb{N}^s \cap (\bigcup_{i=1}^{s}(\alpha_i + C))$ where C is the cone defined by the equations $\langle v, e_i \rangle \geq 0$ for $i = 1, \ldots, n$ and the e_i form the canonical basis of $\mathbb{R}^n$. This follows since $x^{\alpha_i}x^v \in I$ for every $v \in \mathbb{N}^n$.

Theorem 2.4.3. The exponent set of the integral closure of a monomial ideal $I \subset k[x_1, \ldots, x_n]$ is $NP(I) \cap \mathbb{N}^n$. In particular, $NP(\bar{I}) = NP(I)$.

Proof. See [14, p. 11 Proposition 1.4.6]. □

The following proposition justifies the similarity in notation between the Newton polytope of a polynomial and the Newton polyhedron of a monomial ideal.

Proposition 2.4.4. Let $I = (x^{\alpha_1}, \ldots, x^{\alpha_s}) \subset k[x_1, \ldots, x_n]$ and $f_I = x^{\alpha_1} + \cdots + x^{\alpha_s} \in k[x_1, \ldots, x_n]$. Then $NP(I) = NP(f_I) + C$, where $C = \{(u_1, \ldots, u_n) \in \mathbb{R}^n | u_i \geq 0 \text{ for all } i = 1, \ldots n\}$.

Proof. We know $NP(I) = \text{conv}(\bigcup_{i=1}^{m}(\alpha_i + C))$ and $NP(f_I) = \text{conv}(\alpha_1, \ldots, \alpha_m)$. Then:

- $NP(I) \subseteq NP(f_I) + C$: It is clear that $\alpha_i + C \subseteq NP(f_I) + C$ for every $i = 1, \ldots m$, and thus $\bigcup_{i=1}^{m}(\alpha_i + C) \subseteq NP(f_I) + C$. The set $NP(f_I) + C$ is convex and so it must contain the convex hull of any of its subsets.

- $NP(I) \supseteq NP(f_I) + C$: Since $\alpha_i \in NP(I)$ for every $i = 1, \ldots, m$, we have $NP(f_I) \subseteq NP(I)$. By proposition 2.4.2 and theorem 2.4.3 it is clear that $d + C \subseteq NP(I)$ for every $d \in NP(I)$ (addition of lattice points corresponds to multiplication of monomials). In particular $d + C \subseteq NP(I)$ for every $d \in NP(f_I)$ and thus $NP(I) \supseteq NP(f_I) + C$ as desired.

$\square$

Example 2.4.4. Let $I = (xy^5, x^4y^4, x^6y^2) \subset k[x, y]$. Then the exponent set of I, the Newton polytope of f_I and the Newton polyhedron of I are:

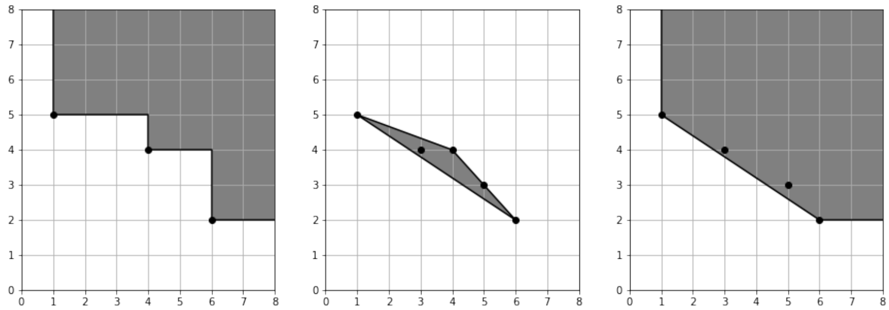


Figure 2.3: The intersection of the dark region in the first graph with $\mathbb{N}^2$ is the exponent set of I and the dots correspond to the generators of I. The second image is the Newton polytope of $f_I = xy^5 + x^4y^4 + x^6y^2$ and the dots are the elements of the intersection $NP(f_I) \cap \mathbb{N}^2$. The third image is the Newton polyhedron of I and the dots correspond to the generators of $\bar{I}$.

Remark 2.4.1. The decomposition $NP(I) = NP(f_I) + C$ is an example of theorem 1.6.3. From remark 2.3.2 we know the Newton polytope of polynomials is closely related to the number of solutions of a polynomial system. We have also seen that this number of solutions is connected to mixed multiplicities. From theorem 2.4.3 one may see that the integral closure of ideals is relevant for the study of multiplicities. This is the topic of the main theorem of the next section.

Remark 2.4.2. The problem of finding the generators of the integral closure of a monomial ideal I is equivalent to solving certain problems in integer programming and thus there are several softwares that can compute $\bar{I}$.

2.5 Reductions

Definition 2.5.1. Let R be a Noetherian ring and $J \subseteq I$ ideals of R. We say J is a reduction of I if there exists an integer n such that $I^{n+1} = JI^n$. Note that if J is a reduction of I, then there exists an integer n such that for all $m \geq 1$, $I^{m+n} = J^m I^n$ and thus $I^{m+n} \subseteq J^m$.

The main result that connects reductions to the integral closure of ideals is the following:

Proposition 2.5.1. Let $J \subseteq I$ be ideals of a Noetherian ring R. Then J is a reduction of I if and only if $I \subseteq \bar{J}$. In particular $r \in R$ is an element of $\bar{I}$ if and only if I is a reduction of $I + (r)$.

Proof. See [14, p. 6 Corollary 1.2.5]. $\square$

The theorem below states why the integral closure of ideals is important in the study of Hilbert-Samuel multiplicities:

Theorem 2.5.2. Let $(R, \mathfrak{m})$ be a Noetherian local ring, J, I $\mathfrak{m}$-primary ideals such that $\bar{J} = \bar{I}$ and M a finite R-module. Then $e(I, M) = e(J, M)$.

In particular, $e(I, M) = e(I + (r), M)$ for every $r \in \bar{I}$.

Proof. See [14, p. 226 Proposition 11.2.1]. $\square$

In view of corollary 2.3.8, it is reasonable to expect that mixed multiplicities are invariant under reductions. This is indeed true:

Proposition 2.5.3. Let $(A, \mathfrak{m})$ be either a local ring or a standard $\mathbb{N}$-graded algebra over a field, where $\mathfrak{m}$ is the maximal graded ideal. Let I be an $\mathfrak{m}$-primary ideal and J an arbitrary ideal of A. If I', J' are reductions of I, J, then

$$e_i(I'|J') = e_i(I|J), \quad \text{for } i = 0, \ldots, \deg P_{R(I|J)}.$$

Proof. See [23, Corollary 3.8]. □

Chapter 3

Topology and mixed multiplicities

In this chapter, our focus will be on giving a topological interpretation to some mixed multiplicities.

Throughout this chapter, maps between topological spaces are always assumed to be continuous.

3.1 CW complexes

Before we introduce the topological spaces that are of interest to us, we first set some notation:

- The unit sphere: $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$
- The unit disk: $D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$
- An n-cell e^n is a topological space homeomorphic to the open n-disk:

$$D^n - \partial D^n = \{x \in \mathbb{R}^n : |x| < 1\}$$

Following the above notation, we can define cell complexes:

Definition 3.1.1. We define CW (or cell) complexes inductively:

- Start with a discrete set X^0. The points of X^0 can be seen as 0-cells.
- To define the n-skeleton X^n from the $n-1$-skeleton X^{n-1} inductively, let D^n_α be unit disks and $\varphi_\alpha \colon \partial D^n_\alpha \to X^{n-1}$ be continuous maps where α ranges over Λ. Define X^n as the quotient space of the disjoint union:

$$X^n = X^{n-1} \coprod_{\alpha \in \Lambda} D^n_\alpha$$

by identifying $x \sim \varphi_\alpha(x)$ for $x \in \partial D^n_\alpha$.

After a finite [1] number of steps the above procedure stops. A space X that can be constructed this way is called a *CW complex* or a *cell complex.* The dimension of X is the number m such that $X = X^m$.

Example 3.1.2. Let $G = (V, E)$ be a graph. Let $X^0 = V$. For every edge $e = (i, j) \in E$, consider a disk D^1_e and a map $\varphi_e \colon \partial D^1_e \to X^0$ such that $\varphi_e(0) = i$ and $\varphi_e(1) = j$. The topological space $X = X^1$ is a 1-dimensional CW complex, that is, graphs are CW complexes.

The next example shows that a space may have more than one CW structure.

Example 3.1.3. The unit sphere S^n has the following CW structure: $X^0 = \{x\}$ and a single n-cell attached via the constant map $\varphi(y) = x$ for every $y \in \partial D^n$. This is the homeomorphism $S^n \cong D^n/\partial D^n$. It is also possible to give a CW structure for the unit sphere S^n by considering an "equator" and attaching two n-cells to it, which can be seen as "north" and "south" hemispheres.

[1] It is also possible to consider infinite dimensional CW complexes. In this case, $X = \bigcup_i X^i$ and X is given the weak topology, that is, $A \subset X$ is open if and only if $A \subset X^i$ is open for every i. For more details see [10].

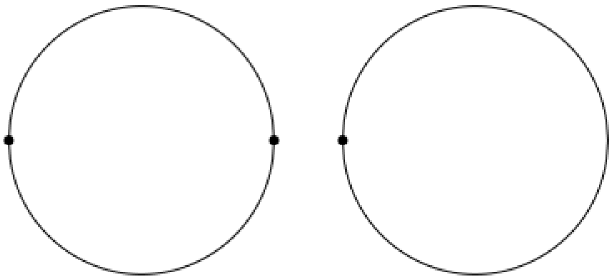


Figure 3.1: Two CW structures of S^1.

There are many other important examples of CW complexes, such as the real (or complex) projective space and compact orientable surfaces in $\mathbb{R}^3$. For more details see [10, Chapter 0].

3.2 Some topological preliminaries

In this section we define the notion of homotopy and singular simplexes, which will be needed in the next section.

Definition 3.2.1. Let X, Y be topological spaces. A *homotopy* from X to Y is a continuous map given by:

$$F\colon X \times [0,1] \to Y, \qquad F(x,t) = f_t(x).$$

Two maps $f, g\colon X \to Y$ are said to be homotopic if there exists a homotopy F such that $f_0 = f$ and $f_1 = g$.

Definition 3.2.2. Let X, Y be topological spaces and $f\colon X \to Y$ a map. We say f is a *homotopy equivalence* if there exists a map $g\colon Y \to X$ such that $f \circ g$ is homotopic to the identity in Y and $g \circ f$ is homotopic to the identity in X. In this case, X and Y are said to be *homotopy equivalent*.

Saying two spaces are homotopy equivalent is clearly weaker than saying two spaces are homeomorphic. Later on in this chapter it will become clear the importance of this notion.

Lastly we define singular simplexes:

Definition 3.2.3. Let $v_0 \dots v_m \in \mathbb{R}^n$ be such that $v_1 - v_0, \dots, v_m - v_0$ are linearly independent. We say the convex hull of $v_0, \dots, v_m$ is a simplex and we denote it by $[v_0, \dots, v_m]$. Note that we implicitly ordered the vertices of this simplex. A face of a simplex is the convex hull of a subset of its vertices. We will denote by $[v_0, \dots, \hat{v}_i, \dots, v_m]$ the simplex that is the convex hull of every vertice of the original simplex except for v_i. We will always assume that the ordering of the vertices of this subsimplex is the same as the ordering of the original simplex. If $v_0 = 0$ and $v_1, \dots, v_n$ is the canonical basis of $\mathbb{R}^n$, then we denote $[v_0, \dots, v_n]$ by Δ^n and call it the standard n-simplex.

Definition 3.2.4. Let X be a topological space. A singular simplex in X is a map $\sigma\colon \Delta^n \to X$.

Let $C_n(X)$ be the free abelian group generated by the set of singular n-simplices in X. Elements of $C_n(X)$ are called n-chains. The map $\partial_n\colon C_n(X) \to C_{n-1}(X)$ defined by:

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_m]}$$

is called the *boundary* map. Note that we are implicitly identifying Δ^{n-1} with $[v_0, \dots, \hat{v}_i, \dots, v_m]$.

The boundary maps give rise to a chain complex of abelian groups:

$$C_\bullet(X)\colon \dots \longrightarrow C_n(X) \longrightarrow C_{n-1}(X) \longrightarrow \cdots \longrightarrow C_1(X) \longrightarrow C_0(X) \longrightarrow 0$$

The homology groups of this complex are called the *singular homology* groups of X and are denoted by $H_n(X)$.

Proposition 3.2.1. Let $X = \{p\}$ then $H_i(X) = 0$ for every $i > 0$ and $H_0(X) = \mathbb{Z}$.

Proof. Since for every n there is only one map $\Delta^n \to X$, by definition it is clear that:

$$C_\bullet(X)\colon \ldots \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \cdots \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

The definition of the boundary map implies ∂_n is an alternating sum of equal terms, since Δ^n has $n+1$ vertices we conclude $\partial_n = 0$ for odd n. Similarly, ∂_n is an isomorphism if n is even.

Computing $H_n(X) = \dfrac{\ker \partial_n}{\mathrm{im}\partial_{n+1}} = 0$ for $n > 0$, and for $n = 0$ we have $H_0(X) = \dfrac{\ker \partial_0}{\mathrm{im}\partial_1} = \mathbb{Z}$. □

As follows from the proposition above, if X is a point, then $H_0(X) \cong \mathbb{Z}$. Since it would be helpful for a point to have trivial homology groups, we define the following augmented chain complex:

$$\tilde{C}_\bullet(X)\colon \ldots \longrightarrow C_n(X) \longrightarrow C_{n-1}(X) \longrightarrow \cdots \longrightarrow C_1(X) \longrightarrow C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0.$$

Where $\epsilon\Big(\sum_i n_i\sigma_i\Big) = \sum_i n_i$. The homology groups of $\tilde{C}.(X)$ are called the *reduced homology* groups of X, and we denote them by $\tilde{H}_n(X) = \dfrac{\ker\ \partial_n}{\mathrm{im}\partial_{n+1}}$ for $n > 0$ and $\tilde{H}_0(X) = \dfrac{\ker\ \varepsilon}{\mathrm{im}\partial_1}$. Note that $H_i(X) \cong \tilde{H}_i(X)$ for $i > 0$.

Definition 3.2.5. Let X be a CW complex. The rank of the i-th homology group of X is called the i-th betti number of X, and is denoted by $b_i(X)$. Similarly, the rank of the i-th reduced homology group of X will be denoted by $\tilde{b}_i(X)$. The alternating sum of the betti numbers of X is called the *Euler characteristic of X* and is denoted by $\chi(X)$.

Example 3.2.6. Using the `Macaulay2` package `SimplicialComplexes` we can compute the reduced homology groups of the real projective plane: $\tilde{H}_i(X) = 0$ for every $i \neq 1$ and $\tilde{H}_1(X) = \mathbb{Z}/2\mathbb{Z}$.

Remark 3.2.1. As can be seen by the definition, it may be hard to compute homology groups for more complicated spaces, since the abelian groups $C_n(X)$ may not be finitely generated. There are other homology theories that can be defined and are easier to compute, such as simplicial[1] and cellular homology. CW complexes play a very important role in the study of the latter.

The last result of this section is a formula relating the homology groups of $X \times Y$ and the homology groups of X and Y.

Definition 3.2.7. Let X be a CW complex and k a field. The homology vector spaces of the chain complex $C.(X;k) := C.(X) \otimes k.$ denoted by $H_i(X;k)$ are called the *homology vector spaces of X with coefficients in k.*

Note that if k has characteristic zero, then the dimension of $H_n(X;k)$ is the i-th betti number of X.

Theorem 3.2.2 (Künneth formula)**.** Let X, Y be CW complexes and k a field. Then

$$\bigoplus_i (H_i(X;k) \otimes H_{n-i}(Y;k)) \cong H_n(X \times Y;k)$$

for all n.

Proof. See [10, Section 3.B]. $\square$

[1] This is the one used by `SimplicialComplexes`.

Proposition 3.2.3. $H_0(S^1) \cong H_1(S^1) \cong \mathbb{Z}$ and $H_i(S^1) = 0$ for $i > 1$.

Proof. See [10]. $\square$

Corollary 3.2.4. Let $\mathbb{T}^n = \underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}}$ denote the n-dimensional torus. The betti numbers of $\mathbb{T}^n$ are the binomial coefficients $\binom{n}{i}$.

Proof. Since $H_0(S^1; \mathbb{Q}) \cong H_1(S^1; \mathbb{Q}) \cong \mathbb{Q}$ and $H_i(S^1; \mathbb{Q}) = 0$ for $i > 1$, the result follows from an inductive argument and the theorem above. $\square$

Proposition 3.2.5. Let X, Y be homotopy equivalent spaces. Then their homology groups are isomorphic. In particular, since $\mathbb{C}^*$ is homotopy equivalent to S^1, the betti numbers of $(\mathbb{C}^*)^n$ are the binomial coefficients $\binom{n}{i}$.

Proof. See [10, p. 111 Theorem 2.10] for the first statement. The second statement follows from the corollary above and the fact that the circle S^1 is homotopy equivalent to $\mathbb{R}^2\backslash\{0\}$. $\square$

3.3 The topology of projective hypersurfaces

In algebraic topology, it is possible to understand the topology of many spaces by studying some of their invariants, such as their fundamental group and their homology groups. One particular example is $\mathbb{R}^n\backslash\{x_1, \dots, x_n\}$, that is, the euclidean space without a finite set of points. Although it is a simple exercise to compute such invariants for this example, if we notice that $\{x_1, \dots, x_n\}$ is an algebraic set, two more interesting questions come up:

- Is it possible to determine the singular homology groups of the zeros of an ideal in $\mathbb{R}[x_1, \dots, x_n]$ (resp. $\mathbb{C}[x_1, \dots, x_n]$) in $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) and describe them only using tools from commutative algebra and algebraic geometry?

- Is it possible to determine the singular homology groups of the complement of the zeros of an ideal in $\mathbb{R}[x_1, \ldots, x_n]$ (resp. $\mathbb{C}[x_1, \ldots, x_n]$) in $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) and describe them only using tools from commutative algebra and algebraic geometry?

The main result of this section is a positive answer to a specific case of the projective version of the questions above, before we state it we need one more definition:

Definition 3.3.1. Given two topological spaces X, Y the wedge sum $X \vee Y$ is the quotient of their disjoint union by identifying a point in X with a point in Y.

A bouquet of spheres is the wedge sum of spheres.

Next we set some notation. Let h be a nonconstant homogeneous polynomial in $\mathbb{C}[z_0, \ldots, z_n]$. Let J_h denote the Jacobian ideal of h, that is, the ideal generated by the partial derivatives of h. Set

$$V(h) = \{p \in \mathbb{P}^n \mid h(p) = 0\}$$
$$D(h) = \{p \in \mathbb{P}^n \mid h(p) \neq 0\}.$$

Definition 3.3.2. A sufficiently general flag of linear subspaces is an increasing chain of subsets of $\mathbb{P}^n$:

$$\mathbb{P}^0 \subset \mathbb{P}^1 \subset \cdots \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n.$$

where each $\mathbb{P}^{n-i}$ is the intersection of the zeros of i general (in the sense of definition 2.3.4) linear forms in $\mathbb{C}[z_0, \ldots, z_n]$.

Theorem 3.3.1. Fix a sufficiently general flag of linear subspaces:

$$\mathbb{P}^0 \subset \mathbb{P}^1 \subset \cdots \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n.$$

For $i = 0, \ldots, n$, set $V(h)_i = V(h) \cap \mathbb{P}^i$ and $D(h)_i = D(h) \cap \mathbb{P}^i$, also set $D(h)_{-1} = \emptyset$ and $V(h)_{-1} = \emptyset$. Then

- $D(h)_i$ is homotopy equivalent to a CW complex obtained from $D(h)_{i-1}$ by attaching $e_{(n-i,i)}(\mathfrak{m}|J_h)$ cells of dimension i. In particular

$$e_{(n-i,i)}(\mathfrak{m}|J_h) = (-1)^i \chi(D(h)_i \backslash D(h)_{i-1}).$$

- $V(h)_i \backslash V(h)_{i-1}$ is homotopy equivalent to a bouquet of $e_{(n-i,i)}(\mathfrak{m}|J_h)$ spheres of dimension $i-1$. In particular,

$$e_{(n-i,i)}(\mathfrak{m}|J_h) = \tilde{b}_{i-1}(V(h)_i \backslash V(h)_{i-1}).$$

Example 3.3.3. Let h be a nonconstant homogeneous polynomial in $S = \mathbb{C}[z_0, \ldots, z_n]$. Set $h = \prod_{i=1}^k g_i^{m_i}$, where the g_i are distinct irreducible factors of h and $m_i \geq 1$. Let $\sqrt{h}$ be the radical $\prod_{i=1}^k g_i$ and d be the degree of $\sqrt{h}$. Set $\mathfrak{m}$ as the ideal $(z_0, \ldots, z_n)$. Applying corollary 2.3.8 we see that $e_{(n,0)}(\mathfrak{m}|J_h) = e(\mathfrak{m}, S) = 1$ and for sufficiently general constants $c_0, \ldots, c_n \in \mathbb{C}$,

$$e_{(n-1,1)}(\mathfrak{m}|J_h) = e(\mathfrak{m}, S/\sum_{j=0}^n c_j \sum_{i=1}^k m_i g_1^{m_1} \ldots g_i^{m_i-1} \ldots g_k^{m_k} \frac{\partial g_i}{\partial z_j} : J_h^\infty).$$

Now let $u = \prod g_i^{m_i-1}$, it is clear that $J_h^i = u^i I^i$ for some ideal I of S and every $i \in \mathbb{N}$. Let $\hat{g}_i$ mean the omission of the factor g_i. Since

$$(\sum_{j=0}^n c_j \sum_{i=1}^k m_i g_1 \ldots \hat{g}_i \ldots g_k \frac{\partial g_i}{\partial z_j}) \supset (u \sum_{j=0}^n c_j \sum_{i=1}^k m_i g_1 \ldots \hat{g}_i \ldots g_k \frac{\partial g_i}{\partial z_j}),$$

we conclude

$$(\sum_{j=0}^n c_j \sum_{i=1}^k m_i g_1 \ldots \hat{g}_i \ldots g_k \frac{\partial g_i}{\partial z_j} : J_h^\infty) \supseteq (u \sum_{j=0}^n c_j \sum_{i=1}^k m_i g_1 \ldots \hat{g}_i \ldots g_k \frac{\partial g_i}{\partial z_j} : J_h^\infty).$$

On the other hand, let $r \in (\sum_{j=0}^{n} c_j \sum_{i=1}^{k} m_i g_1 \dots \hat{g}_i \dots g_k \frac{\partial g_i}{\partial z_j} : J_h^{\infty})$. By definition, there exists $n \in \mathbb{N}$ such that $rJ_h^n \subset (\sum_{j=0}^{n} c_j \sum_{i=1}^{k} m_i g_1 \dots \hat{g}_i \dots g_k \frac{\partial g_i}{\partial z_j})$. In particular,

$$rJ_h^{n+1} = u(ru^n I^n)I \subset (u \sum_{j=0}^{n} c_j \sum_{i=1}^{k} m_i g_1 \dots \hat{g}_i \dots g_k \frac{\partial g_i}{\partial z_j})$$

which implies

$$(\sum_{j=0}^{n} c_j \sum_{i=1}^{k} m_i g_1 \dots \hat{g}_i \dots g_k \frac{\partial g_i}{\partial z_j} : J_h^{\infty}) = (\sum_{j=0}^{n} c_j \sum_{i=1}^{k} m_i g_1^{m_1} \dots g_i^{m_i - 1} \dots g_k^{m_k} \frac{\partial g_i}{\partial z_j} : J_h^{\infty})$$

and thus

$$e_{(n-1,1)}(\mathfrak{m}|J_h) = e(\mathfrak{m}, S/\sum_{j=0}^{n} c_j \sum_{i=1}^{k} m_i g_1 \dots \hat{g}_i \dots g_k \frac{\partial g_i}{\partial z_j} : J_h^{\infty}).$$

Next, we want to prove that

$$e(\mathfrak{m}, S/\sum_{j=0}^{n} c_j \sum_{i=1}^{k} m_i g_1 \dots \hat{g}_i \dots g_k \frac{\partial g_i}{\partial z_j} : J_h^{\infty}) = e(\mathfrak{m}, S/\sum_{j=0}^{n} c_j \sum_{i=1}^{k} m_i g_1 \dots \hat{g}_i \dots g_k \frac{\partial g_i}{\partial z_j}).$$

Let $v = \sum_{j=0}^{n} c_j \sum_{i=1}^{k} m_i g_1 \dots \hat{g}_i \dots g_k \frac{\partial g_i}{\partial z_j}$. From the short exact sequence

$$0 \longrightarrow \frac{(v : J_h^{\infty})}{v} \longrightarrow \frac{S}{v} \longrightarrow \frac{S}{(v : J_h^{\infty})} \longrightarrow 0$$

it is clear that if $\dim\, (v : J_h^{\infty})/v < n$, then $\dim\, S/v = \dim\, S/(v : J_h^{\infty}) = n$ and the multiplicities are equal. Moreover, since $\dim M = \dim \operatorname{Supp}(M)$ for finite S-modules, we only need to prove $\operatorname{ht} \mathfrak{p} \geq 2$ for every $\mathfrak{p} \in \operatorname{Supp}((v : J_h^{\infty})/v)$. Equivalently, we need to prove that $(v : J_h^{\infty})_{\mathfrak{p}} = (v)_{\mathfrak{p}}$ for every $\mathfrak{p} \in \operatorname{Spec} S$ such that $\operatorname{ht} \mathfrak{p} = 1$.

For any ideal $I \subset S$, let $V(I)$ denote $\{\mathfrak{p} \in \operatorname{Spec} S \mid \mathfrak{p} \supseteq I\}$. By the equality

$$V(J_h) = V(J_{\sqrt{h}}) \cup V(g_1^{m_1-1}) \cup \cdots \cup V(g_k^{m_k-1})$$

and since every prime ideal in $V(J_{\sqrt{h}})$ has height at least 2, the only prime ideals of ht $\mathfrak{p} = 1$ of S that contain J_h are the prime ideals that contain one of the $g_i^{m_i}$. Note that:

1. If $\mathfrak{p} = g_i$ and $g_i \not| \, v$, then $(v)_{g_i} = (1)$ and $(v : J_h^\infty)_{g_i} = (v)_{g_i}$.

2. If $\mathfrak{p} \not\supset J_h$, then $(J_h)_\mathfrak{p}^i = (1)_\mathfrak{p}$ for every $i \in \mathbb{N}$ and thus $(v : J_h^\infty)_\mathfrak{p} = (v)_\mathfrak{p}$.

In particular, since $J_{\sqrt{h}} \not\subset (g_i)$ for any i, by prime avoidance there exists an element $v \in J_{\sqrt{h}}$ such that $(v) \not\subset \bigcup_i (g_i)$. In other words, there exists an element v of $J_{\sqrt{h}}$ such that g_i does not divide v for any i. Since $\mathbb{C}$ is infinite, we can choose constants $c_0, \ldots, c_n$ so that $v = \sum_{j=0}^n c_j \sum_{i=1}^k m_i g_1 \ldots \hat{g}_i \ldots g_k \frac{\partial g_i}{\partial z_j}$. From the remarks above we conclude $(v : J_h^\infty)_\mathfrak{p} = (v)_\mathfrak{p}$ for every prime ideal of height 1 of S and thus:

$$\begin{aligned}
e_{(n-1,1)}(\mathfrak{m}|J_h) &= e(\mathfrak{m}, S/\sum_{j=0}^{n} c_j \sum_{i=1}^{k} m_i g_1^{m_1} \ldots g_i^{m_i-1} \ldots g_k^{m_k} \frac{\partial g_i}{\partial z_j} : J_h^\infty) \\
&= e(\mathfrak{m}, S/\sum_{j=0}^{n} c_j \sum_{i=1}^{k} m_i g_1 \ldots \hat{g}_i \ldots g_k \frac{\partial g_i}{\partial z_j} : J_h^\infty) \\
&= e(\mathfrak{m}, S/\sum_{j=0}^{n} c_j \sum_{i=1}^{k} m_i g_1 \ldots \hat{g}_i \ldots g_k \frac{\partial g_i}{\partial z_j}) \\
&= d - 1.
\end{aligned}$$

Chapter 4

Graphs, matroids and hyperplane arrangements

Our goal for this chapter is to connect the characteristic polynomials of matroids with the characteristic polynomials of hyperplane arrangements and the chromatic polynomials of graphs. These polynomials are the main objects of the main theorem of this text.

4.1 Hyperplane arrangements

Definition 4.1.1. A hyperplane arrangement $\mathcal{A}$ is a finite set of hyperplanes in some vector space $V \cong k^n$ where k is a field. Throughout this text, the field k will be either $\mathbb{R}$ or $\mathbb{C}$, unless stated otherwise.

Moreover, if $\bigcap\limits_{A \in \mathcal{A}} A \neq \emptyset$ we say $\mathcal{A}$ is central. If every hyperplane in $\mathcal{A}$ is a vector subspace we say $\mathcal{A}$ is linear.

Since hyperplanes are algebraic sets, for each hyperplane $A \in \mathcal{A}$ there exists a linear form L_A such that $A = \{a \in k^n | L_A(a) = 0\} = V(L_A)$. The polynomial

$Q_{\mathcal{A}} = \prod_{A \in \mathcal{A}} L_A$ is called the *defining polynomial of* $\mathcal{A}$.

Definition 4.1.2. Let $\mathcal{A}$ be a hyperplane arrangement in $V \cong k^n$. The dimension of $\mathcal{A}$ is the dimension of V. The rank of $\mathcal{A}$ denoted by $\text{rank}(\mathcal{A})$ is the dimension of the space spanned by the normals of the hyperplanes in $\mathcal{A}$.

Given an arrangement $\mathcal{A}$ in $V \cong k^n$, we can define two important arrangements related to $\mathcal{A}$. If $\mathcal{A}$ is linear, we can consider $\mathcal{A}$ as the union of hyperplanes in $\mathbb{P}_k^{n-1}$ by taking a hyperplane $H \in \mathcal{A}$ as the hyperplane at infinity. The *decone of* $\mathcal{A}$ *with respect to* H is the arrangement $\bar{\mathcal{A}}^H$ of the hyperplanes of $\mathcal{A}$ in $k^{n-1} = \mathbb{P}_k^{n-1} \backslash H$. If $\mathcal{A}$ is a central arrangement, we define the arrangement $c\mathcal{A}$ which we call the *cone of* $\mathcal{A}$ by its defining polynomial. Let $L_1(\mathbf{x}) - a_1, \ldots, L_m(\mathbf{x}) - a_m$ be the irreducible factors of the defining polynomial $Q_{\mathcal{A}}$ of $\mathcal{A}$. The defining polynomial of the cone of $\mathcal{A}$ is the polynomial $y \prod_{i=1}^{m}(L_i - a_i y)$.

Definition 4.1.3. Let $\mathcal{A}$ be an arrangement in V, and let $L(\mathcal{A})$ denote the set of all nonempty intersections of hyperplanes in $\mathcal{A}$, including V (which can be seen as the intersection of an empty set of hyperplanes). Define a partial order in $L(\mathcal{A})$ by the relation: $x \leq y$ in $L(\mathcal{A})$ if $x \supseteq y$ (that is, reverse inclusion in V). The poset $L(\mathcal{A})$ with reverse inclusion is called the *intersection poset* of $\mathcal{A}$. Note that the element $V \in L(\mathcal{A})$ satisfies $V \leq x$ for all $x \in L(\mathcal{A})$.

Next we set some notation:

Given a poset P with a partial order relation $\leq$, if $x \leq y$ then the (closed) interval $[x, y]$ is the set $\{z \in P | x \leq z \leq y\}$. We denote by $\hat{0}$ an element in a poset P such that $\hat{0} \leq x$ for every $x \in P$. Such an element need not exist.

If P is a poset and $x, y \in P$ are such that $x < y$ and there are no $z \in P$ satisfying $x < z < y$ then we say y *covers* x and denote it by $x \lessdot y$.

Definition 4.1.4. A *chain of length* k in a poset P is a set $x_0 < x_1 < \cdots < x_k$ of elements of P. The chain is said to be saturated if $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k$. If every maximal chain of P has length n, we say P is *graded of rank n*. The rank of an arrangement will be denoted by rank(P). We can then define a rank function $\mathrm{rk}\colon P \to \mathbb{N}$ as follows:

- $\mathrm{rk}(x) = 0$ if x is a minimal element of P.
- $\mathrm{rk}(y) = \mathrm{rk}(x) + 1$ if $x \lessdot y$ in P.

Moreover, if $x < y$, we write $\mathrm{rk}(x, y) = \mathrm{rk}(y) - \mathrm{rk}(x)$ for the *length* of the inverval $[x, y]$.

Proposition 4.1.1. Let $\mathcal{A}$ be an arrangement in a vector space $V \cong k^n$. Then the intersection poset $L(\mathcal{A})$ is graded of rank equal to rank$(\mathcal{A})$.

Proof. See [22, p. 8 Proposition 1.1]. □

4.2 The characteristic polynomial of an arrangement

In this section we introduce the characteristic polynomial of hyperplane arrangements. As we will see on later chapters, these polynomials are closely related to mixed multiplicities.

Definition 4.2.1. A poset P is *locally finite* if every interval $[x, y]$ is finite. The set of all closed intervals of P will be denoted by Int(P). If $f\colon \mathrm{Int}(P) \to \mathbb{Z}$ is a function, we write $f(x, y)$ for $f([x, y])$.

Definition 4.2.2. Let P be a locally finite poset. The *Möbius function* of P denoted by $\mu = \mu_P$ is a function $\mu\colon \mathrm{Int}(P) \to \mathbb{Z}$ satisfying:

- $\mu(x, x) = 1$ for every $x \in P$.

- $\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z)$ for all $x < y \in P$.

Moreover, if P has an element $\hat{0}$, then we write $\mu(x)$ for $\mu(\hat{0}, x)$.

Next we define the characteristic polynomial of an arrangement $\mathcal{A}$:

Definition 4.2.3. The *characteristic polynomial* of an arrangement $\mathcal{A}$ is the sum:

$$\chi_{\mathcal{A}}(t) = \sum_{x \in L(\mathcal{A})} \mu(x) t^{\dim(x)}.$$

Note that it follows directly from the definition that $\chi_{\mathcal{A}}(t) = t^n - \#\mathcal{A}t^{n-1} + \ldots$, where $\mathcal{A}$ is an arrangement in $V \cong k^n$.

Example 4.2.4. Let $h = xyzw(x + y + z + w) \in \mathbb{C}[x, y, z, w]$ be the defining polynomial of a hyperplane arrangement $\mathcal{A}$. The decone of $\mathcal{A}$ with respect to x is defined by the polynomial $yzw(1 + z + w + y)$. Then $\chi_{\bar{\mathcal{A}}^H}(t) = t^3 - 4t^2 + 6t - 4$.

Example 4.2.5. Let $h = xyzw \in \mathbb{C}[x, y, z, w]$ be the defining polynomial of an arrangement $\mathcal{A}$. The decone of $\mathcal{A}$ with respect to x is defined by the polynomial yzw. Then $\chi_{\bar{\mathcal{A}}^H}(t) = \binom{3}{0}t^3 - \binom{3}{1}t^2 + \binom{3}{2}t - \binom{3}{3} = (t - 1)^{4-1}$. More generally, if $h = x_1 \ldots x_n \in \mathbb{C}[x_1, \ldots, x_n]$ defines the hyperplane arrangement $\mathcal{B}$, it is possible to prove that $\chi_{\mathcal{B}}(t) = (t - 1)^n$. For more details see [22, Proposition 1.2, p. 10].

Example 4.2.6. Let $h = \prod_{1 \leq i < j \leq n} (x_i - x_j) \in \mathbb{C}[x_1, \ldots, x_n]$. The arrangement $\mathcal{B}_n$ in $\mathbb{C}^n$ defined by h is called the *braid arrangement*. The characteristic polynomial of this arrangement is: $\chi_{\mathcal{B}_n}(t) = t(t-1)\ldots(t-n+1)$. This arrangement will play an important role on the next section.

Theorem 4.2.1 (Whitney)**.** Let $\mathcal{A}$ be an arrangement in an n-dimensional vector space. Then

$$\chi_{\mathcal{A}}(t) = \sum_{\substack{\mathcal{B} \subseteq \mathcal{A}, \\ \mathcal{B} \text{ central}}} (-1)^{\#\mathcal{B}} t^{n - \operatorname{rank}(\mathcal{B})}.$$

Proof. See [22, p. 17, Theorem 2.4]. □

4.3 The chromatic polynomial

Let $G = ([n], E)$ be a simple graph. We associate to the graph G the following subarrangement of the braid arrangement:

$$x_i - x_j = 0, \qquad (i, j) \in E.$$

We will denote the arrangement above by $\mathcal{A}_G$ and call it the *graphical arrangement of* G. Note that the braid arrangement is simply the graphical arrangement of the complete graph K_n.

As we will see on the next theorem, the arrangement $\mathcal{A}_G$ is closely related to some invariants of the graph G.

Definition 4.3.1. A coloring of a graph $G = (V, E)$ is a map $\kappa \colon V \to \mathbb{N}$ such that $\kappa(v_1) \neq \kappa(v_2)$ for every $(v_1, v_2) \in E$. Let m be a positive integer. Denote by $\chi_G(m)$ the number of colorings κ such that the image of κ is a subset of $[m]$. It is possible to prove that the function χ_G is a polynomial and thus we call it the *chromatic polynomial* of G.

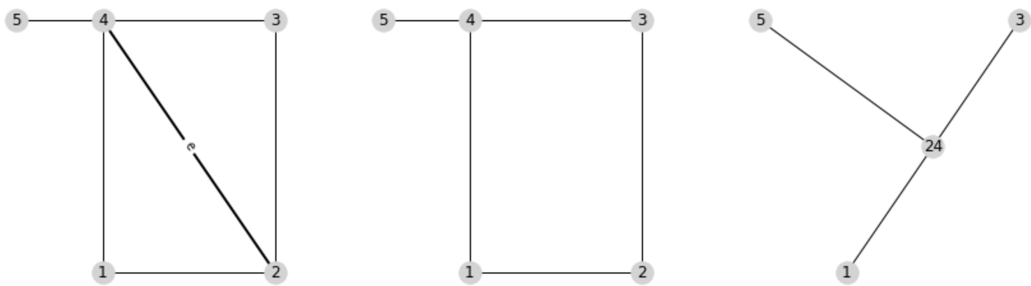
Theorem 4.3.1. For any graph G, the following equality holds:

$$\chi_G(t) = \chi_{\mathcal{A}_G}(t).$$

Proof. See [22, p. 25, Theorem 2.7]. □

Next we state a lemma that is used in a proof of the theorem above. This lemma can be used as a way of computing chromatic polynomials of graphs.

Definition 4.3.2. Let $G = (V, E)$ be a graph and $e \in E$. We denote by $G - e$ the graph $(V, E - \{e\})$, that is, G with the edge e deleted. We denote by G/e the graph obtained from G by contracting the edge $e = (i, j)$, that is, by considering the vertices i, j as a new vertex ij such that every edge that is incident to either i or j is incident to ij.



(a) G (b) $G - e$ (c) G/e

Lemma 4.3.2 (Deletion-contraction)**.** Let $G = (V, E)$ be a graph and e an edge of G. Then

$$\chi_G(t) = \chi_{G-e}(t) - \chi_{G/e}(t).$$

Proof. Let $e = (i, j)$. The lemma follows directly from the following remarks:

Colorings of G are in bijection with colorings of $G - e$ such that the value of i and j are not equal. Colorings of G/e are in bijection with colorings of $G - e$ such that i and j have the same value. □

From the lemma above it is possible to implement a recursive algorithm that given a graph G, returns its chromatic polynomial. It is also possible to prove that the chromatic polynomial is indeed a polynomial using the lemma above.

Example 4.3.3. Let

$$h = (z_1 - z_2)(z_1 - z_4)(z_2 - z_3)(z_2 - z_4)(z_3 - z_4)(z_4 - z_5) \in \mathbb{C}[z_1, z_2, z_3, z_4, z_5].$$

Let $\mathcal{A}_G$ be the graphic arrangement defined by h and G the graph associated to the graphical arrangement. Using the `SageMath` programming language we can compute the characteristic polynomial of $\mathcal{A}_G$, $\chi_{\mathcal{A}_G}(t) = t^5 - 6t^4 + 13t^3 - 12t^2 + 4t$. Implementing the algorithm mentioned above we conclude $\chi_G(t) = \chi_{\mathcal{A}_G}(t)$.

4.4 Matroids

Let V be a finite dimensional vector space with $n = \dim V$. It is clear that a set of linearly independent vectors of V satisfy the properties below:

1. If $J \subset V$ is a set of linearly independent vectors, then every subset J' of J is also a set of linearly independent vectors.

2. Let J_1, J_2 be maximal sets of linearly independent vectors of V. Then $|J_1| = |J_2|$.

In this section, we introduce objects that generalize the ideas above. Such objects are called *Matroids*. One great achievement of Matroid theory is the fact that matroids can be defined in a variety of ways. For this reason two (equivalent) definitions of matroids are given.

Definition 4.4.1. Let E be a finite set and I a collection of subsets of E satisfying the three properties below:

1. $\emptyset \in I$.

2. If $I_1 \in I$ and $I_2 \subseteq I_1$ then $I_2 \in I$.

3. If I_1 and I_2 are in I and $|I_1| < |I_2|$, then there is an element $e \in I_2 \backslash I_1$ such that $I_1 \cup \{e\} \in I$.

The ordered pair $M = (E, I)$ is called a *matroid*. The sets in I are called *independent sets*.

Definition 4.4.2. Let $M = (E, C)$ be an ordered pair where C is a collection of subsets of the finite set E satisfying the following properties:

1. $\emptyset \notin C$.

2. If $C_1 \in C$ and $C_2 \subset C_1$ then $C_2 \notin C$.

3. If $C_1, C_2 \in C$ and $x \in C_1 \cap C_2$, then there exists $C' \subseteq C_1 \cup C_2 \backslash \{x\}$ such that $C' \in C$.

Elements of C are called *circuits*, and M is said to be a matroid.

A circuit C' can be thought of as a minimal dependent set, that is, a dependent (not independent) set such that every subset of C' is independent. For a proof of the equivalence of the definitions above (and other definitions) see [19, Chapter 1]. By $x \in M = (E, I)$ we mean $x \in E$.

Given two matroids $M_1 = (E_1, I_1)$ and $M_2 = (E_2, I_2)$ we say M_1 and M_2 are isomorphic if there exists a bijection $f \colon E_1 \to E_2$ such that $f(i_1) \in I_2$ for every $i_1 \in I_1$. The isomorphism will be denoted by $M_1 \cong M_2$.

Example 4.4.3. Let k be a field and A an $r \times s$ matrix with coefficients in k. Let $E = [s]$ and define I as follows:

$I' \subset E \in I \iff$ the columns indexed by the elements of I' are linearly independent.

Then $M[A] = (E, I)$ is a matroid.

A matroid M isomorphic to $M[A]$ for some matrix A over a field k is said to be *representable over* k. The matrix A is said to be a *representation of* M *over* k. Two natural questions arise from this definition:

- Does there exist a matroid that is not representable over any field?
- Is it possible for a matroid to be representable over a field k_1 but not over a field k_2?

The following examples are answers to the questions above.

Example 4.4.4. Let E be the set of all nonzero vectors in $\dfrac{\mathbb{Z}}{2\mathbb{Z}}^3 = \mathbb{F}_2^3$ and I be defined by usual linear independence. The matroid $F_7 = (E, I)$ is called the *Fano matroid*. It is representable over $\mathbb{F}_2$ but not over $\mathbb{R}$.

Example 4.4.5. Let $E = [8]$ and I be the set of all subsets of E of cardinality ≤ 4 except for:

$$\{1,2,3,4\}, \{1,2,5,6\}, \{3,4,5,6\}, \{3,4,7,8\}, \{5,6,7,8\}.$$

The matroid $V_8 = (E, I)$ is called the *Vámos matroid*. This matroid is not representable over any field.

Next we define a rank function on the subsets of a matroid $M = (E, I)$. Let $T \subseteq E$, we define the *rank of* T as

$$\mathrm{rk}(T) = \max\{|J| : J \in I \text{ and } J \subseteq T\}.$$

A maximal subset of rank k is called a *k-flat* of M. Let $L(M)$ denote the poset of flats of M ordered by inclusion. Note that $\mathrm{rk}(\emptyset) = 0$ and $\mathrm{rk}(M) := \mathrm{rk}(E) = n$ for some positive n. The poset $L(M)$ is graded of rank n and has an element $\hat{0}$.

We are now ready to define the characteristic polynomial of a matroid M.

Definition 4.4.6. Let $M = (E, I)$ be a matroid of rank n (that is, $\mathrm{rk}(M) = n$). Following notation from section 4.2, we define the polynomial

$$\chi_M(t) = \sum_{x \in L(M)} \mu(x) t^{n-\mathrm{rk}(x)}$$

which we call the *characteristic polynomial of M*.

The next theorem shows a connection between central arrangements and representable matroids.

Theorem 4.4.1. Let $\mathcal{A}$ be a central arrangement in a k-vector space $V \cong k^n$. Let E denote the set $\{\mathfrak{n}_H | H \in \mathcal{A}\}$ where $\mathfrak{n}_H$ is the normal vector of the hyperplane $H \in \mathcal{A}$. Let A be the matrix such that its columns are the vectors in E. Then $M[A] = M_{\mathcal{A}}$ is a matroid and $L(M_{\mathcal{A}}) \cong L(\mathcal{A})$, that is, there exists an order-preserving bijection between $L(M_{\mathcal{A}})$ and $L(\mathcal{A})$.

Proof. See [22, p. 35 Proposition 3.6]. □

One important consequence of the isomorphism of posets above is that their möbius and rank functions are "equal", that is, if f is an order-preserving bijection between $L(M_{\mathcal{A}})$ and $L(\mathcal{A})$, then $\mu(f(x)) = \mu(x)$ and $\mathrm{rk}(x) = \mathrm{rk}(f(x))$ for every $x \in L(\mathcal{A})$.

Proposition 4.4.2. Let $\mathcal{A}$ be an arrangement in a vector space $V \cong k^n$. The rank function of $L(\mathcal{A})$ is given by $\mathrm{rk}(x) = n - \dim x$ for every $x \in L(\mathcal{A})$.

Proof. See [22, p. 8 Proposition 1.1]. □

In particular, we can rewrite the characteristic polynomial of an n-dimensional arrangement $\mathcal{A}$ as

$$\chi_{\mathcal{A}}(t) = \sum_{x \in L(\mathcal{A})} \mu(x) t^{n-\mathrm{rk}(x)}.$$

We can thus conclude that if $M_{\mathcal{A}}$ has rank r, then

$$\begin{aligned}
t^{n-r}\chi_{M_{\mathcal{A}}}(t) &= t^{n-r}\sum_{x\in L(M_{\mathcal{A}})}\mu(x)t^{r-\mathrm{rk}(x)}\\
&= \sum_{x\in L(M_{\mathcal{A}})}\mu(x)t^{n-\mathrm{rk}(x)}\\
&= \chi_{\mathcal{A}}(t).
\end{aligned}$$

The last connection we need to establish in this chapter is between graphs and matroids. We have already defined *representable* matroids, a class of matroids closely related to hyperplane arrangements. Next we define another class of matroids that is directly connected to graphs called *graphic* matroids.

Definition 4.4.7. Let $G = (V, E)$ be a graph (G can have parallel edges). Let C be the subsets of E that are cycles of G. Then $M_G = (E, C)$ is a matroid where C are the circuits of M_G. If a matroid M is isomorphic to M_G for some graph G we say that M is a *graphic* matroid.

Theorem 4.4.3. Graphic matroids are representable over every field.

Proof. See [19, p. 135 Lemma 5.1.3]. □

Given a graph $G = (V, E)$, define $D(G) = (V', E')$ as a directed graph such that $V' = V$ and the edges of $D(G)$ are the edges of G with a direction arbitrarily assigned. Then let $A = [a_{ij}]$ denote the incidence matrix of $D(G)$, that is, a $|V'|$ by $|E'|$ matrix such that

$$a_{ij} = \begin{cases} 1, & i \text{ is the head of the edge } j\\ -1, & i \text{ is the tail of the edge } j\\ 0, & \text{otherwise.}\end{cases}$$

The matroid M_A is a representation of M_G.

From previous results in this section and the remarks above we conclude $\chi_G(t) = \chi_{\mathcal{A}_G}(t) = t^q\chi_{M_G}(t)$ for some integer $q \geq 0$. The next theorem gives an interpretation of the number q in terms of the graph G.

Theorem 4.4.4. Let G be a graph with q connected components. Then

$$\chi_G(t) = t^q\chi_{M_G}(t).$$

Proof. The result follows since the constant coefficient of χ_M is nonzero for a matroid M and the multiplicity of 0 as a root of χ_G is the number of connected components of G. For more details see [19]. □

Example 4.4.8. Let G be the graph defined in example 4.3.3. The incidence matrix of G (with respect to the orientation below) is

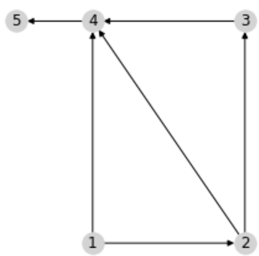


Figure 4.2: An orientation of the graph G in example 4.3.3.

$$
\begin{array}{cc}
\begin{array}{cccccc} 12 & 24 & 23 & 14 & 45 & 34 \end{array} & \\
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 \\
-1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 \\
0 & -1 & 0 & -1 & 1 & -1 \\
0 & 0 & 0 & 0 & -1 & 0
\end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\end{array}
$$

Using the `Macaulay2` package `Matroids` to compute the characteristic polynomial of M_G we verify

$$\chi_G(t) = t^5 - 6t^4 + 13t^3 - 12t^2 + 4t = t(t^4 - 6t^3 + 13t^2 - 12t + 4) = t\chi_{M_G}(t).$$

Example 4.4.9. Let A be the 5 by 5 identity matrix. Then the characteristic polynomial of M_A is $\chi_{M_A}(t) = (t-1)(\binom{4}{0}t^4 - \binom{4}{1}t^3 + \binom{4}{2}t^2 - \binom{4}{3}t + \binom{4}{4})$. Compare this example with example 2.2.5.

Example 4.4.10. Let A be the following matrix:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix}.$$

Then $\chi_{M_A}(t) = t^4 - 5t^3 + 10t^2 - 10t + 4 = (t-1)(t^3 - 4t^2 + 6t - 4)$. Compare this example with example 2.2.6.

Chapter 5

Log-concave sequences and examples

In this brief chapter we introduce log-concave sequences and give some examples of interesting sequences.

5.1 Log-concavity and unimodality

Let $A = (a_0, \ldots, a_n)$ be a sequence of real numbers. The sequence is called *log-concave* if

$$a_{i-1}a_{i+1} \leq a_i^2$$

for all $i = 1, \ldots, n-1$. If the subsequence of nonzero elements of A has only consecutive elements of A, we say A has no *internal zeros*. We say a sequence $(a_0, \ldots, a_n)$ is *sign-alternating* if $(-1)^i a_i > 0$ for every $i = 0, \ldots, n$.

If a sequence of real numbers $(a_0, \ldots, a_n)$ satisfies: $a_0 \leq \cdots \leq a_k \geq \cdots \geq a_n$ we say the sequence is *unimodal*.

Definition 5.1.1. Let $A = (a_0, \ldots, a_n)$ and $B = (b_0, \ldots, b_m)$ be sequences of real numbers. Let $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{i=0}^{m} b_i x^i$. The *convolution* of A and B denoted by $A * B$ is the sequence of coefficients of $fg(x) = \sum_{i=0}^{n+m} c_i x^i$.

Next we state two results that are very useful.

Proposition 5.1.1. If A and B are nonnegative log-concave sequences with no internal zeros, then so is $A * B$.

Proof. See [21, Proposition 2]. $\square$

Proposition 5.1.2. If $A = (a_0, \ldots, a_n)$ is a nonnegative log-concave sequence with no internal zeros, then A is unimodal.

Proof. By log-concavity we know $\frac{a_{i+1}}{a_i} \leq \frac{a_i}{a_{i-1}}$ for every $i = 1, \ldots, n-1$. From the ratio $\frac{a_i}{a_{i-1}}$ we can say if the next element of the sequence is smaller or bigger than the last one, and since the ratios form a decreasing sequence, the sequence must be unimodal. $\square$

5.2 Betti tables: an interesting example

In this section we introduce the graded betti numbers of an ideal. From these numbers we get interesting examples of unimodal sequences that are not log-concave.

Let $I = (f_1, \ldots, f_s)$ be a homogeneous ideal of the $\mathbb{N}$-graded polynomial ring $R = k[x_1, \ldots, x_n]$. The map $\varphi\colon R(-\deg f_1) \oplus \cdots \oplus R(-\deg f_s) \to R$ sending the canonical basis to the generators of I is a homogeneous map and so it has a homogeneous kernel and we can thus proceed with this process. The resulting complex of the construction just described is called the *minimal graded free resolution of I*.

Example 5.2.1. Let $R = k[x_1, \ldots, x_7]$ and $I = (x_3x_4, x_3x_7, x_1x_7, x_2x_4, x_1x_3)$. The minimal graded free resolution of I is

$$0 \longrightarrow R(-4) \oplus R(-5) \longrightarrow R(-3)^5 \oplus R(-4) \longrightarrow R(-2)^5 \longrightarrow R \longrightarrow 0.$$

Proposition 5.2.1. Let R be an $\mathbb{N}$-graded polynomial ring over a field k. The minimal graded free resolution of a homogeneous ideal I of R has finitely many nonzero modules.

Proof. See [3, Section 1.5]. $\square$

Definition 5.2.2. Let R be an $\mathbb{N}$-graded polynomial ring over a field and I a homogeneous ideal. Let

$$C_\bullet : \ldots \longrightarrow 0 \longrightarrow F_m \longrightarrow F_{m-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow R \longrightarrow 0$$

be the minimal graded free resolution of I. Then by definition we know

$$F_i = \bigoplus_j R(-j)^{\beta_{ij}(I)}.$$

The numbers $\beta_{ij}(I)$ are called the *graded betti numbers of I*. The table $\beta(I)$ such that the (i,j)-th entry corresponds to β_{ii+j} is called the *betti table of I*.

Example 5.2.3. Let $R = k[x_1, \ldots, x_7]$ and $I = (x_3x_4, x_3x_7, x_1x_7, x_2x_4, x_1x_3)$. The table below is $\beta(I)$, the dotted entries are zero.

$j \backslash i$	0	1	2	3
0	1	.	.	.
1	.	5	5	1
2	.	.	1	1

It is clear that if I and J are homogeneous ideals such that $\dfrac{R}{I} \cong \dfrac{R}{J}$, then $\beta(I) = \beta(J)$. It is also possible that two ideals have the same betti tables and nonisomorphic quotient rings.

Let G be a graph and $I(G)$ the edge ideal defined in example 1.2.3. One may ask how many tables are the betti table of an edge ideal $I(G)$ in $k[x_1, \ldots, x_n]$ where k is a field of characteristic zero. More specifically, one may ask how many tables are the betti tables of the edge ideal of a graph with a specific property P.

The questions above lead us to examples of unimodal sequences that are not log-concave:

Example 5.2.4. Let $a_2^n(i)$ denote the number of different betti tables of edge ideals of graphs with n vertices and i edges over a field k of characteristic zero.

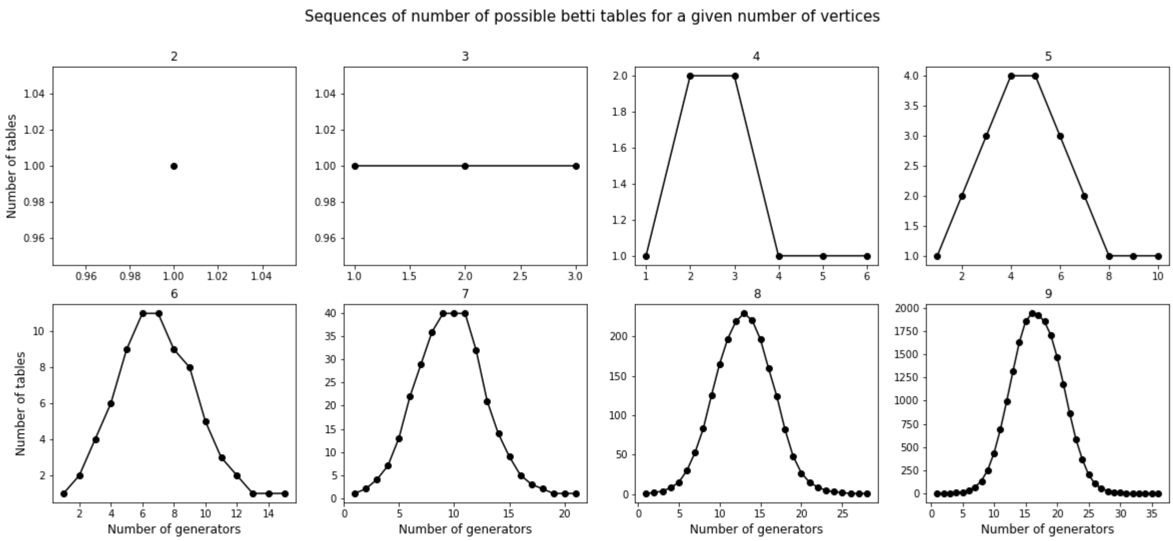


Figure 5.1: Graphs of the sequences $\{a_2^j(i)\}_i$ for $j = 2, \ldots, 9$.

As it is clear from the graphs, the sequences $\{a_2^j(i)\}_i$ are unimodal but not log-concave for $j = 4, \ldots, 9$.

Example 5.2.5. Given a betti table β we say it has *size* (p, r) if every (i, j)-th entry with $i > p$ and $j > r$ is zero and if (p', r') is another tuple satisfying the same property, then $p' \geq p$ and $r' \geq r$. The size of the betti table in example 5.2.3 is $(3, 2)$.

Similarly to example 5.2.4, let $b_2^j(i)$ denote the number of different betti table sizes of edge ideals of graphs with n vertices and i edges over a field k of characteristic zero.

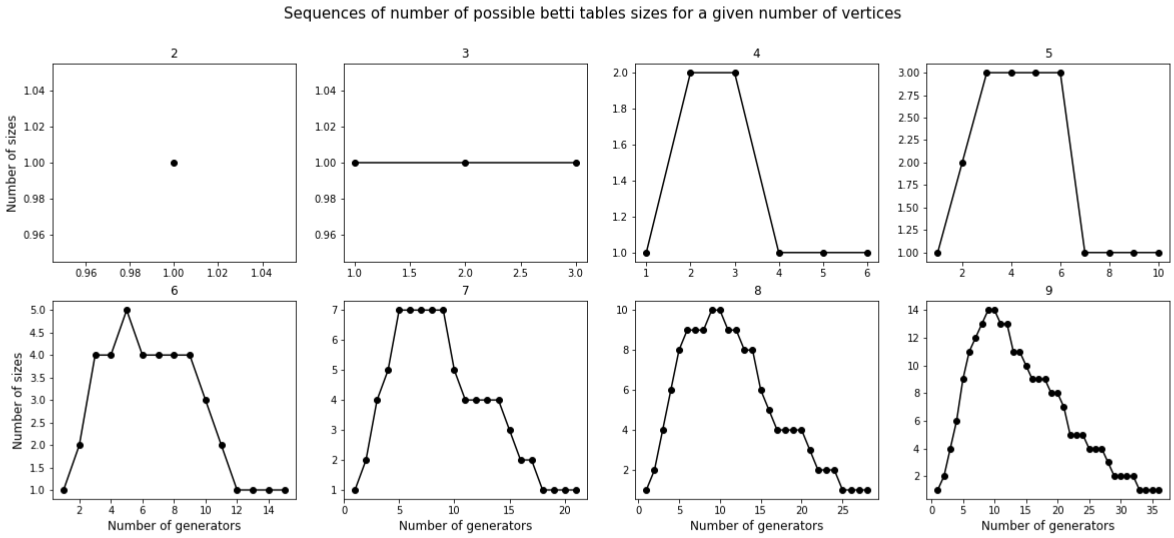


Figure 5.2: Graphs of the sequences $\{b_2^j(i)\}_i$ for $j = 2, \ldots, 9$.

These sequences are also unimodal but not log-concave for $j = 4, \ldots, 9$.

From the examples above it is natural to ask if the sequences $\{a_2^j(i)\}_i$ and $\{b_2^j(i)\}_i$ are unimodal for every j. It is possible to define the sequences $\{a_n^j(i)\}_i$ and $\{b_n^j(i)\}_i$ where instead of edge ideals we consider square-free monomial ideals generated by monomials of the same degree n[1]. One can then ask if the sequences $\{a_n^j(i)\}_i$ and $\{b_n^j(i)\}_i$ are unimodal for every n and j.

Remark 5.2.1. The computations in this chapter were made using the graph isomorphism testing program `Nauty` and `Macaulay2`.

[1]These ideals are sometimes called the *facet* ideals of pure simplicial complexes.

Chapter 6

Representable homology classes of projective spaces

In this chapter we introduce the object from algebraic geometry that connects the combinatorial objects introduced in chapter 4 with certain mixed multiplicities.

Throughout this chapter, $\mathbb{P}^n$ will denote the projective space over the algebraically closed field K.

6.1 Algebraic cycles

Throughout this section the ring R will always be the standard $\mathbb{N}^2$-graded ring $K[x_0, \ldots, x_n, y_0, \ldots, y_m]$ such that $\deg x_i = (1, 0)$ and $\deg y_j = (0, 1)$ for every $i = 0, \ldots, n$ and $j = 0, \ldots, m$ unless stated otherwise.

Let A be a standard $\mathbb{N}^r$-graded K-algebra and let A_+ be its maximal homogeneous ideal. The Proj of A denoted by Proj$A$ is the set of all homogeneous prime ideals of A that do not contain A_+.

Given a prime ideal $\mathfrak{p} \in \text{Proj}R$, we set:

$$\mathcal{Z}(\mathfrak{p}) := \{p \in \mathbb{P}^n \times \mathbb{P}^m | f(p) = 0 \;\; \forall f \in \mathfrak{p}\}.$$

Definition 6.1.1. Let $X = \mathbb{P}^n \times \mathbb{P}^m$ and let $Z(X)$ denote the free abelian group generated by $\mathcal{Z}(\mathfrak{p})$ for every $\mathfrak{p} \in \mathrm{Proj}R$. We call $Z(X)$ the group of *algebraic cycles* of X. The group $Z(X)$ is graded by codimension, that is, let $Z(X)^r$ be the subgroup of $Z(X)$ generated by $\mathcal{Z}(\mathfrak{p})$ for every $\mathfrak{p} \in \mathrm{Proj}R$ such that height $\mathfrak{p} = r$ then $Z(X) = \bigoplus_{r \in \mathbb{N}} Z(X)^r$. An element Y of $Z(X)$ is called a cycle, if $Y \in Z(X)^r$ then Y is called an r-cycle.

Let I be a homogeneous ideal of R that does not contain R_+. By theorem 1.2.4 we know there are prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ and primary ideals $q_1, \ldots, q_s$ such that q_i is $\mathfrak{p}_i$-primary for every $i = 1, \ldots, s$ and

$$I = \bigcap_{i=0}^{s} q_i.$$

Since I is homogeneous we can assume the $\mathfrak{p}_i$ and q_i are homogeneous as well. By the second part of theorem 1.2.4 and corollary 1.2.3 the ring $S_{\mathfrak{p}} = \dfrac{R_{\mathfrak{p}}}{IR_{\mathfrak{p}}}$ is Artinian for every minimal associated prime $\mathfrak{p}$, thus the length $l_{\mathfrak{p}} = \ell(S_{\mathfrak{p}})$ is finite.

We can then define the cycle

$$\langle I \rangle = \sum_{\mathfrak{p} \in \min \mathrm{Ass}(R/I)} l_{\mathfrak{p}} \mathcal{Z}(\mathfrak{p}) \in Z(X).$$

6.2 Rational equivalence and the Chow group

In this section we define the subgroup of rationally equivalent cycles $\mathrm{Rat}(X)$. We are interested in the quotient group $Z(X)/\mathrm{Rat}(X)$.

Definition 6.2.1. Let $\Phi \subseteq \mathbb{P}^n \times \mathbb{P}^m$, then the ideal of Φ is $\mathcal{I}(\Phi) = \{f \in R | f(p) = 0 \;\; \forall p \in \Phi\}$.

Definition 6.2.2. Let $\mathfrak{q} \in \text{Proj}R[z_0, z_1]$ where $R[z_0, z_1]$ is a standard $\mathbb{N}^3$-graded ring. Set $\mathcal{Z}(\mathfrak{q}) = \{p \in \mathbb{P}^n \times \mathbb{P}^m \times \mathbb{P}^1 | f(p) = 0 \quad \forall f \in \mathfrak{q}\}$. Given $t_0 \in \mathbb{P}^1$, we can consider the cycle $\langle \mathcal{Z}(\mathfrak{q}) \cap \mathbb{P}^n \times \mathbb{P}^m \times \{t_0\}\rangle := \langle \mathcal{I}(\{p \in \mathbb{P}^n \times \mathbb{P}^m | [p, t_0] \in \mathcal{Z}(\mathfrak{q}) \cap \mathbb{P}^n \times \mathbb{P}^m \times \{t_0\}\})\rangle$.

Let $\text{Rat}(\mathbb{P}^n \times \mathbb{P}^m)$ be the subgroup of $Z(\mathbb{P}^n \times \mathbb{P}^m)$ generated by differences of the form

$$\langle \mathcal{Z}(\mathfrak{q}) \cap \mathbb{P}^n \times \mathbb{P}^m \times \{t_0\}\rangle - \langle \mathcal{Z}(\mathfrak{q}) \cap \mathbb{P}^n \times \mathbb{P}^m \times \{t_1\}\rangle$$

where $t_0, t_1 \in \mathbb{P}^1$ and $\mathfrak{q}$ is any ideal in $\text{Proj}R[z_0, z_1]$ such that $\mathcal{Z}(q)$ is not contained in $\mathbb{P}^n \times \mathbb{P}^m \times \{t_0\}$ for any $t_0 \in \mathbb{P}^1$.

The group $A(\mathbb{P}^n \times \mathbb{P}^m) = Z(\mathbb{P}^n \times \mathbb{P}^m)/\text{Rat}(\mathbb{P}^n \times \mathbb{P}^m)$ is called the *Chow group* of $\mathbb{P}^n \times \mathbb{P}^m$. Since $\text{Rat}(\mathbb{P}^n \times \mathbb{P}^m)$ is graded by codimension, the Chow group of $\mathbb{P}^n \times \mathbb{P}^m$ is also graded by codimension. If $Y \in Z(\mathbb{P}^n \times \mathbb{P}^m)$ is a cycle, we write $[Y]$ for its equivalence class. Similarly if I is a homogeneous ideal of R, we write $[I]$ for the equivalence class of $\langle I\rangle$.

Remark 6.2.1. The definition of the Chow group of $\mathbb{P}^n \times \mathbb{P}^m$ is in some ways similar to the homology groups defined in section 3.2. In both definitions a specific equivalence relation is defined and the groups of interest are the quotients of the groups of cycles by this equivalence relation.

Remark 6.2.2. It is clear that we can also define the Chow group of $\mathbb{P}^n$ by replacing the ring R by the standard $\mathbb{N}$-graded ring $K[x_0, \ldots, x_n]$. The definition of Chow group is in fact even more general, but these are the only two cases we will need. For more details see [6, Chapter 1].

Theorem 6.2.1. The following isomorphisms of abelian groups hold:

- $A(\mathbb{P}^n) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$, where $\alpha \in A(\mathbb{P}^n)^1$ is the rational equivalence class of a hyperplane.

- $A(\mathbb{P}^n \times \mathbb{P}^m) \cong \mathbb{Z}[\alpha, \beta]/(\alpha^{n+1}, \beta^{m+1})$, where $\alpha^r \beta^s$ is the rational equivalence class of $[\mathbb{P}^{n-r} \times \mathbb{P}^{m-s}]$.

In particular, every cycle in $A(\mathbb{P}^n \times \mathbb{P}^m)$ can be written as

$$\sum_{i \in [n], j \in [m]} e_{ij}[\mathbb{P}^{n-i} \times \mathbb{P}^{m-j}].$$

where the e_{ij} are integers.

Proof. See [6, p. 51 Theorem 2.10] and [6, p. 44 Theorem 2.1]. □

Given an element $\xi \in A(\mathbb{P}^n \times \mathbb{P}^m)^k$, by the theorem above we know

$$\xi = \sum_i e_i[\mathbb{P}^{k-i} \times \mathbb{P}^i]$$

where $e_i = 0$ if $n < k - i$ or $m < i$.

Definition 6.2.3. We say an element $\xi \in A(\mathbb{P}^n \times \mathbb{P}^m)^k$ is *representable* if there exists a homogeneous ideal I of $K[x_0, \ldots, x_n, y_0, \ldots, y_m]$ such that $[I] = \xi$.

From the last theorem and the definition above one can ask the following question:

- For what sequences e_i is the element $\xi = \sum_i e_i[\mathbb{P}^{k-i} \times \mathbb{P}^i] \in A(\mathbb{P}^n \times \mathbb{P}^m)^k$ representable?

The answer to the question above is the following theorem:

Theorem 6.2.2. Write $\xi \in A(\mathbb{P}^n \times \mathbb{P}^m)^k$ as an integral linear combination

$$\xi = \sum_i e_i[\mathbb{P}^{k-i} \times \mathbb{P}^i]$$

where the term containing e_i is zero if $n < k - i$ or $m < i$.

- If ξ is an integer multiple of either

$$[\mathbb{P}^n \times \mathbb{P}^m], [\mathbb{P}^n \times \mathbb{P}^0], [\mathbb{P}^0 \times \mathbb{P}^m], [\mathbb{P}^0 \times \mathbb{P}^0],$$

  then ξ is representable if and only if the integer is 1.

- Otherwise, some positive integer multiple of ξ is representable if and only if the e_i form a nonzero log-concave sequence of nonnegative integers with no internal zeros.

Proof. See [13, Section 5]. $\square$

6.3 Multidegrees

In this section we introduce the notions of degree and multidegree, as will soon become clear, together with theorem 6.2.2 the concept of multidegree plays a very important role in the proof of the main theorems of this text. Throughout this section the field K will be algebraically closed.

Definition 6.3.1. Let R be the $\mathbb{N}$-graded ring $K[x_0, \ldots, x_n]$ and let I be a homogeneous ideal of R. The set $\mathcal{Z}(I) := \{p \in \mathbb{P}^n | f(p) = 0 \;\; \forall f \in I\}$ is called a *variety*. If the radical of I is prime, then $\mathcal{Z}(I)$ is said to be irreducible. The ring $R/\sqrt{I}$ is called the *coordinate ring of* $\mathcal{Z}(I)$, where $\sqrt{I}$ denotes the radical of I. The dimension of $\mathcal{Z}(I)$ is the Krull dimension of its coordinate ring.

Definition 6.3.2. Let $X \subset \mathbb{P}^n$ be an irreducible k-dimensional variety. Let A be a general $(n-k)$-plane, that is, the zero locus of k general linear forms. Then the degree of X denoted by $\deg X$ is the number of points in the intersection $A \cap X$.

Theorem 6.3.1. Let X be an irreducible variety and S its coordinate ring. Then the degree of X is the multiplicity of S.

Proof. See [8, Lecture 18]. □

Note that we can extend the definitions in definition 6.3.1 to the product of two projective spaces by replacing R with the $\mathbb{N}^2$-graded ring $K[x_0, \ldots, x_n, y_0, \ldots, y_m]$.

We have already seen how mixed multiplicities generalize the notion of multiplicity. Our next goal is to introduce the geometric invariant that generalizes definition 6.3.2 and theorem 6.3.1.

Definition 6.3.3. Let $X \subset \mathbb{P}^n \times \mathbb{P}^m$ be a k-dimensional variety. Let $Y \subset \mathbb{P}^n$ be a general $(n-i)$-plane (that is, the zero locus of i general linear forms in $K[x_0, \ldots, x_n]$) and $Z \subset \mathbb{P}^m$ a general $(m-k+i)$-plane. The number

$$d_i = |(Y \times Z) \cap X|$$

is called the i-th multidegree of X.

We are interested in a particular case of the definition above:

Let $X \subset \mathbb{P}^n$ be an irreducible variety, S its coordinate ring, $h_0, \ldots, h_m \in S$ be homogeneous elements of the same degree and consider $J = (h_0, \ldots, h_m)$. Since the h_i are not all zero, we can consider the map:

$$\varphi_J \colon \begin{array}{c} U \to \mathbb{P}^m \\ [x_0 : \cdots : x_n] \mapsto [h_0(x_0, \ldots, x_n) : \cdots : h_m(x_0, \ldots, x_n)] \end{array}$$

where U is a dense open subset of X such that the h_i do not simultaneously vanish at any point of U.

Let Γ_{φ_J} denote the closure of the graph of φ_J, that is, the closure of

$$\{[u, \varphi_J(u)] \,|u \in U\} \subset \mathbb{P}^n \times \mathbb{P}^m.$$

Following this notation we have the following results:

Proposition 6.3.2. The coordinate ring of Γ_{φ_J} is $R(\mathfrak{m}|J) = \bigoplus_{(u,v)\in\mathbb{N}^2} \frac{\mathfrak{m}^u J^v}{\mathfrak{m}^{u+1} J^v}$ where $\mathfrak{m}$ is the maximal homogeneous ideal of S.

Proof. See [5, Section 5.2]. □

Theorem 6.3.3. Let k be the dimension of Γ_{φ_J}. Then the following equalities hold in $A(\mathbb{P}^n \times \mathbb{P}^m)^k$:

$$[\Gamma_{\varphi_J}] = \sum_j d_j [\mathbb{P}^j \times \mathbb{P}^{k-j}] = \sum_i e_{(n-i,i)}(\mathfrak{m}|J)[\mathbb{P}^{k-i} \times \mathbb{P}^i].$$

In particular, $d_{n-i} = e_{(n-i,i)}(\mathfrak{m}|J)$ for every i.

Proof. See [6, Section 2.1.7]. □

From theorem 6.3.3 and theorem 6.2.2 we get the following result:

Corollary 6.3.4. If J is an ideal of a standard graded domain over an algebraically closed field generated by elements of the same degree, then the mixed multiplicities of $\mathfrak{m}$ and J form a log-concave sequence of nonnegative integers with no internal zeros.

Using the `Macaulay2` package `Cremona` we can compute the multidegrees of Γ_{φ_J} for a given J.

Example 6.3.4. Let $h = xyzw(x + y + z + w)$ and let J_h denote the ideal of $\mathbb{C}[x, y, z, w]$ generated by the partial derivatives of h. Since every partial derivative of h has the same degree, we can define the map $\varphi = \varphi_{J_h}$ and compute the multidegrees of Γ_φ: $d_3 = 1, d_2 = 4, d_1 = 6, d_0 = 4$.

Example 6.3.5. Let $h = xyzw$ and let J_h denote the ideal of $\mathbb{C}[x, y, z, w]$ generated by the partial derivatives of h. Then the multidegrees of the closure of the graph of φ_{J_h} are $d_3 = 1, d_2 = 3, d_1 = 3, d_0 = 1$.

Example 6.3.6. Let $h = (z_1 - z_2)(z_1 - z_4)(z_2 - z_3)(z_2 - z_4)(z_3 - z_4)(z_4 - z_5)$ be an element of $\mathbb{C}[z_1, \ldots, z_5]$. Then the multidegrees of the closure of the graph of φ_{J_h} are $d_4 = 1, d_3 = 5, d_2 = 8, d_1 = 4, d_0 = 0$. Compare this example with example 4.3.3 and note that

$$t^5 - 6t^4 + 13t^3 - 12t^2 + 4t = (t - 1)(t^4 - 5t^3 + 8t^2 - 4t).$$

If we consider $h' = z_6 h \in \mathbb{C}[z_1, \ldots, z_6]$, then the multidegrees of the closure of $\varphi_{J_{h'}}$ are $d_5 = 1, d_4 = 6, d_3 = 13, d_2 = 12, d_1 = 4, d_0 = 0$.

Chapter 7

Log-concavity of characteristic polynomials

Throughout this text we introduced mixed multiplicities, betti numbers and the characterisitc polynomials of hyperplane arrangements and matroids. In this chapter we state the final results needed for the proof of the conjectures.

7.1 Characteristic polynomials and mixed multiplicities

Our goal in this section is to state the theorem that connects certain mixed multiplicities to the coefficients of the characterisitc poylnomial of a matroid representable over $\mathbb{C}$. In order to do so, we need the following theorem relating the characteristic polynomial of an arrangement $\mathcal{A}$ to the characteristic polynomial of the decone of $\mathcal{A}$ with respect to some $H \in \mathcal{A}$.

Theorem 7.1.1. Let $\mathcal{A}$ be a linear arrangement in $\mathbb{C}^{n+1}$ and H a hyperplane in $\mathcal{A}$. Let $\bar{\mathcal{A}}^H \subset \mathbb{C}^n$ denote the decone of $\mathcal{A}$ with respect to H.

Then the following equality holds:

$$\chi_{\mathcal{A}}(t) = (t-1)\chi_{\bar{\mathcal{A}}^H}(t).$$

Proof. See [22, p. 52 Corollary 4.8]. $\square$

Remark 7.1.1. Theorem 7.1.1 is a particular case of the *Modular element factorization theorem.* For more details see [22, p. 49 Theorem 4.13]

We are finally ready to state the last result we need for the conjecture over $\mathbb{C}$.

Theorem 7.1.2. Let $h \in \mathbb{C}[z_0, \ldots, z_n]$ be the defining polynomial of a linear arrangement $\mathcal{A} \subset \mathbb{C}^{n+1}$ and let $H \in \mathcal{A}$. Let $\mathfrak{m} = (z_0, \ldots, z_n)$ and $J_h \subset \mathbb{C}[z_0, \ldots, z_n]$ be the ideal generated by the partial derivatives of h. Then the following equalities hold:

$$\chi_{\bar{\mathcal{A}}^H}(t) = \sum_{i=0}^{n}(-1)^i b_i(D(h))t^{n-i} = \sum_{i=0}^{n} e_{(n-i,i)}(\mathfrak{m}|J_h)t^{n-i}.$$

Proof. For the first equality see [20]. For the second equality see [13, Corollary 25]. $\square$

Example 7.1.1. Here are some examples throughout the text that are explained by theorem 7.1.2.

- See example 2.2.5, example 4.2.5 and example 6.3.5 for different ways of computing the invariants for $h = xyzw \in \mathbb{C}[x, y, z, w]$ and more generally $h = z_0 \ldots z_n \in \mathbb{C}[z_0, \ldots, z_n]$.
- See example 2.2.6, example 4.2.4 and example 6.3.4 for $h = xyzw(x+y+z+w) \in \mathbb{C}[x, y, z, w]$.
- See example 4.3.3 and example 6.3.6 for $h = (z_1 - z_2)(z_1 - z_4)(z_2 - z_3)(z_2 - z_4)(z_3 - z_4)(z_4 - z_5) \in \mathbb{C}[z_1, \ldots, z_5]$.

Theorems 7.1.1, 7.1.2, 6.2.2 and proposition 5.1.1 imply:

Theorem 7.1.3. Let M be a matroid representable over $\mathbb{C}$. Then the coefficients of $\chi_M(t)$ form a sign-alternating log-concave sequence of integers with no internal zeros.

Proof. Let $\mathcal{A} \subset \mathbb{C}^{n+1}$ be a linear arrangement that represents M and let $H \in \mathcal{A}$. By theorem 7.1.2 and theorem 6.2.2 the coefficients of $\chi_{\bar{\mathcal{A}}^H}$ form a sign-alternating log-concave sequence with no internal zeros. The result follows by theorem 7.1.1 and proposition 5.1.1. □

In particular, by theorem 4.4.3 and theorem 4.3.1 we also have the following result:

Corollary 7.1.4. Let G be a graph. Then the coefficients of the chromatic polynomial of G form a sign-alternating log-concave sequence of integers with no internal zeros.

In his paper [13], Huh also proved the following generalization of theorem 7.1.3:

Theorem 7.1.5. Let M be a matroid representable over any field of characteristic zero. Then the coefficients of $\chi_M(t)$ form a sign-alternating log-concave sequence of integers with no internal zeros.

Proof. See [13, Corollary 27]. □

Later on Huh and Katz proved a generalization of theorem 7.1.5:

Theorem 7.1.6. [15] Let M be a representable matroid. Then the coefficients of $\chi_M(t)$ form a sign-alternating log-concave sequence of integers with no internal zeros.

And finally in 2018 Adiprasito, Huh and Katz proved log-concavity for every matroid:

Theorem 7.1.7. [16] Let M be a matroid. Then the coefficients of $\chi_M(t)$ form a sign-alternating log-concave sequence of integers with no internal zeros.

Appendices

Appendix A

A proof of theorem 3.3.1

A.1 Integral closure of rings

In previous chapters we introduced the integral closure of ideals. In this section we give a brief review of the integral closure of rings. The main result in this section will be used in the proof of theorem 3.3.1.

The concept of integral closure for rings generalizes the notion of algebraic closure for fields.

Definition A.1.1. Let R be a ring and S an R-algebra such that $R \subset S$. An element $s \in S$ is said to be *integral over* R if there exists $n \in \mathbb{N}$ and $r_1, \ldots, r_n \in R$ such that

$$s^n + s^{n-1}r_1 + \cdots + r_n = 0.$$

The set of all elements of S that are integral over R is called the *integral closure of* R *in* S and will be denoted by $\bar{R}_S$ or just $\bar{R}$.

If $\bar{R} = R$ we say R is integral over S.

Proposition A.1.1. Let R be a ring and S an R-algebra that contains R. Then

$\bar{R}$ is a ring.

Proof. See [14, p. 26, Corollary 2.1.11]. □

The following criterion is very useful:

Proposition A.1.2. Let $R \subseteq S$ be an inclusion of rings and let $x_1, \ldots, x_n \in S$. Then $x_1, \ldots, x_n$ are integral over R if and only if $R[x_1, \ldots, x_n]$ is a finitely generated R-submodule of S.

Proof. See [14, p. 26, Lemma 2.1.9]. □

Note that the criterion above is similar to proposition 2.5.1. The following results will be used in the proof of theorem 3.3.1.

Proposition A.1.3. Let k be a field and $(R, \mathfrak{m})$ a standard $\mathbb{N}$-graded k-algebra where $\mathfrak{m}$ is the maximal homogeneous ideal, J, I ideals in R, $J \subseteq I$, and B the subalgebra of $\mathcal{F}_I(R)$ generated over $R/\mathfrak{m}$ by $(J + \mathfrak{m}I)/\mathfrak{m}I$. Then $J \subseteq I$ is a reduction if and only if $B \subseteq \mathcal{F}_I(R)$ is module-finite.

Proof. See [14, p. 161, Proposition 8.2.4]. □

The subalgebra B defined above is the image of $\mathcal{F}_J(R)$ through the natural map $\varphi : \mathcal{F}_J(R) \to \mathcal{F}_I(R)$.

Proposition A.1.4. Following the notation from above, for any ideal I,

$$\dim \mathcal{F}_I(R) \leq \dim R.$$

Proof. See [14, p. 100, Proposition 5.1.6]. □

A.2 The degree of the jacobian map (base case)

Let h be a nonconstant homogeneous polynomial in $\mathbb{C}[x_0, \ldots, x_n]$ and let φ_{J_h} be the map defined in section 6.3. In this section we introduce the notion of *degree* for the map φ_{J_h}. This is the last definition we need before proving theorem 3.3.1.

Proposition A.2.1. Assume the image of φ_{J_h} is dense in $\mathbb{P}^n$. The number of points in the preimage of a generic point in the image of φ_{J_h} is an invariant of φ_{J_h}. This number is called the *degree of* φ_{J_h} and will be denoted by $\deg(\varphi_{J_h})$. If the image of φ_{J_h} is not dense in $\mathbb{P}^n$, then we define $\deg(\varphi_{J_h}) = 0$.

Proof. See [8, p. 80, Proposition 7.16]. $\square$

Following the definition above, we can state the theorem of [4] that will be used as the base case in the proof of theorem 3.3.1.

Theorem A.2.2. [4, Theorem 1] Let $H \subset \mathbb{P}^n$ be a general hyperplane, $D(h) = \{p \in \mathbb{P}^n | h(p) \neq 0\}$ and $V(h) = \{p \in \mathbb{P}^n | h(p) = 0\}$.

- $D(h)$ is homotopy equivalent to a CW complex obtained from $D(h) \cap H$ by attaching $\deg(\varphi_{J_h})$ cells of dimension n.
- $V(h) \backslash H$ is homotopic to a bouquet of $\deg(\varphi_{J_h})$ spheres of dimension $n - 1$.

Corollary A.2.3. Let $h = \prod_{i=0}^{s} g_i^{m_i}$ and $\sqrt{h} = \prod_{i=0}^{s} g_i$ be polynomials in $\mathbb{C}[x_0, \ldots, x_n]$. Then $\deg(\varphi_{J_h}) = \deg(\varphi_{J_{\sqrt{h}}})$.

Proof. This follows since $\deg(\varphi_{J_h})$ is a topological invariant of $D(h)$ and $D(h) = D(\sqrt{h})$. $\square$

Proposition A.2.4. Let h be a polynomial in $\mathbb{C}[x_0, \ldots, x_n]$. Then $\deg(\varphi_{J_h}) = e_{(0,n)}(\mathfrak{m}|J_h)$.

Proof. By theorem 6.3.3, $e_{(0,n)}(\mathfrak{m}|J_h)$ is the number of points in the preimage of a general point in the image of the jacobian map φ_{J_h}. This is exactly the definition of $\deg(\varphi_{J_h})$. $\square$

A.3 The inductive step

In this last section we use previous results on integral closure and reductions to prove theorem 3.3.1.

Lemma A.3.1. Let x be a nonzero linear form in $S = \mathbb{C}[z_0, \ldots, z_n]$, $\bar{S} = S/xS$ and $J_{\bar{h}}$ be the Jacobian ideal of the class of h in $\bar{S}$. Then, for a sufficiently general x, $J_{\bar{h}}$ is a reduction of $J_h\bar{S}$.

Proof. Assume the partial derivatives of h are linearly independent. Let V be the vector space of linear forms in S, that is, $V = S_1$ and let W be the vector space spanned by the partial derivatives of h. Given a linear form $x = c_0z_0 + \ldots c_nz_n$ we can assume without loss of generality that $c_n \neq 0$. Then $\bar{S}$ is the polynomial ring generated by the classes of $z_0, \ldots, z_{n-1}$. By the chain rule, $J_{\bar{h}}$ is generated by the restrictions of the polynomials

$$c_n\frac{\partial h}{\partial z_i} - c_i\frac{\partial h}{\partial z_n}, \quad 0 \leq i < n.$$

Multiplying the linear form by a constant does not change the quotient ring, this means choosing a linear form x is equivalent to choosing a point in $\mathbb{P}^n$. Since we are assuming $c_n \neq 0$, we are choosing a point in $\mathbb{C}^n = \mathbb{P}^n \backslash H$ where H is the hyperplane $c_n = 0$.

On the other hand, since we are assuming the partial derivatives of h are linearly independent, the space generated by the polynomials above has dimension $\dim W - 1$. In particular, we identified an affine piece of the projective space of lines in V with an affine piece of the projective space of hyperplanes in W.

From the remarks above, we only need to prove that the ideal generated by a sufficiently general subspace of dimension n of $(J_h\bar{S})_{\deg h-1}$ generates a reduction of $J_h\bar{S}$. This follows from the graded Noether normalization theorem applied to $\mathcal{F}_J(S)$, proposition A.1.3 (the extension $\varphi(\mathcal{F}_I(S)) \subseteq \mathcal{F}_J(S)$ is module-finite) and proposition A.1.4 (the dimension of $\mathcal{F}_J(S)$ is at most n). $\square$

Lemma A.3.2. Let $S = \mathbb{C}[z_0, \ldots, z_n]$, x a general linear form and $\bar{S} = S/xS$. Let $\bar{x}$ be the residue class of x, then for any ideal J:

$$P_{R(\mathfrak{m}\bar{S}|J\bar{S})}(u, v) = P_{R(\mathfrak{m}|J)}(u, v) - P_{R(\mathfrak{m}|J)}(u - 1, v).$$

Proof. The same proof of lemma 2.3.4 works. $\square$

Finally we have everything we need to prove theorem 3.3.1:

Proof of theorem 3.3.1. We will prove the result using induction on the number of variables of $S = \mathbb{C}[z_0, \ldots, z_n]$.

If $n = 1$, by example 3.3.3 we know $e_{(1,0)}(\mathfrak{m}|J_h) = 1$ and $e_{(0,1)}(\mathfrak{m}|J_h) = d - 1$ where d is the degree of $\sqrt{h}$, both multiplicities agree with the fact that $D(h)_0$ is a point and $D(h)_1$ is homotopy equivalent to a CW complex obtained from $D(h)_0$ by attaching $d - 1$ cells of dimension 1. Where the latter follows since by theorem 6.3.3 the statements of theorem 3.3.1 (for $i = 1$, $n = 1$) and theorem A.2.2 are the same.

If $n > 1$, let $\bar{S} = S/xS$ where x is a sufficiently general linear form in S and $\bar{h}$ the residue class of h in $\bar{S}$. Since $\bar{S}$ is a polynomial ring in $n - 1$ variables, we are able to apply induction as follows:

By proposition 2.5.3 and lemma A.3.1 $J_{\bar{h}}$ is a reduction of $J_h\bar{S}$ and thus

$$e_{(n-1-i,i)}(\mathfrak{m}\bar{S}|J_{\bar{h}}) = e_{(n-1-i,i)}(\mathfrak{m}\bar{S}|J_h\bar{S}),$$

from lemma A.3.2, for $i < n$ the following equalities hold

$$e_{(n-1-i,i)}(\mathfrak{m}\bar{S}|J_h\bar{S}) = e_{(n-i,i)}(\mathfrak{m}|J_h).$$

Lastly, for $i = n$ we use proposition A.2.4 to conclude $e_{(0,n)}(\mathfrak{m}|J_h) = \deg(\varphi_{J_h})$. The result then follows by induction and theorem A.2.2. $\square$

Appendix B

Polynomial systems and mixed multiplicities

In this chapter we state the main results of [24] in order to study applications of mixed multiplicities in other areas of mathematics. In particular, we prove that a particular system of differential equations that can be used to model two species with a mutual predation dynamic has at most one steady state where both species coexist.

B.1 A sparse version of Bezout's theorem

Let $f_1, \ldots, f_n$ be nonconstant polynomials in $\mathbb{C}[x_1, \ldots, x_n]$. We are interested in finding upper bounds to the number of solutions of the polynomial system $f_1 = ... = f_n = 0$ in $\mathbb{C}^n$.

Theorem B.1.1 (Bézout)**.** Following the above notation, assume $f_1, \ldots, f_n$ have a finite number of common zeros in $\mathbb{C}^n$ and let d_i be the degree of f_i. Then the polynomial system $f_1 = \cdots = f_n = 0$ has at most $d_1 \ldots d_n$ zeros in $\mathbb{C}^n$.

Proof. See [9, p. 52, Theorem 7.7]. $\square$

Remark B.1.1. If we consider homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_n]$ and count the multiplicity of each zero, then we can say the polynomial system has exactly $d_1 \ldots d_n$ zeros in $\mathbb{P}^n$.

Remark B.1.2. Let $F_1, \ldots, F_n$ be homogeneous polynomials in $S = \mathbb{C}[x_0, \ldots, x_n]$ such that $F_i(1, x_1, \ldots, x_n) = f_i(x_1, \ldots, x_n)$ and $\deg F_i = d_i$. Then

$$e((x_0, \ldots, x_n), S/F_i) = d_i.$$

The upper bound in theorem B.1.1 can then be rewritten as a product of multiplicities.

Next we give the refined version of Bezout's theorem for systems of two polynomials in $\mathbb{C}[x, y]$.

Theorem B.1.2 (Bernstein)**.** Let f_1 and f_2 be nonconstant polynomials in $\mathbb{C}[x, y]$. Assume f_1, f_2 have finitely many common zeros in $(\mathbb{C}^*)^2$. Then the number of solutions of the polynomial system $f_1 = f_2 = 0$ in $(\mathbb{C}^*)^2$ is bounded above by:

$$\frac{1}{2}(e(k[NP(f_1 f_2)]) - e(k[NP(f_1)]) - e(k[NP(f_2)])).$$

Proof. See [24, Theorem 3.1]. $\square$

Although the theorem above is a very particular case of a more general theorem, we can already derive interesting applications from it. This is the theme of the next section.

Remark B.1.3. To state a more general version of Bernstein's theorem, we would need to define either the notion of mixed volumes for a sequence of polytopes, or the notion of mixed multiplicities for a sequence of ideals $J_1, \ldots, J_n$. In [24] the authors prove that the mixed volume of polytopes are the mixed multiplicities of certain monomial ideals.

B.2 An application: coexistence of a pair of species

Suppose $P_1(t), P_2(t)$ are two functions that represent the quantities of each species at the time t. Assume that the dynamic between both species is of competition, that is, an encounter between an individual of species 1 and an individual of species 2 is bad for both species. A simple model that can be used to study the possible behaviours of this dynamic is:

$$\begin{cases} \dfrac{dP_1}{dt}(t) = P_1(t)(1 - P_1(t)) - b_1 P_1(t)P_2(t) \\ \dfrac{dP_2}{dt}(t) = P_2(t)(1 - P_2(t)) - b_2 P_1(t)P_2(t) \end{cases}$$

where $b_1, b_2 \in (0, \infty)$ are parameters of the model.

One important information of a model are its steady points, that is, points $(x, y) \in \mathbb{R}^2$ such that if $P_1(t_0) = x, P_2(t_0) = y$ for some $t_0 \in (0, \infty)$, then

$$\frac{dP_1}{dt}(t_0) = \frac{dP_2}{dt}(t_0) = 0.$$

It is clear that the points we are looking for are exactly the zeros of the polynomials $f_1(x, y) = x(1 - x) - b_1 xy$ and $f_2(x, y) = y(1 - y) - b_2 xy$ in $\mathbb{C}[x, y]$.

A particular property of a steady state p that we are looking for is that p has no zero coordinates. We say the species can *coexist* in a model if there exists a steady state $p \in (0, \infty)^n$.

Example B.2.1. Following the notation above, let $b_1 = 0.5$, $b_2 = 0.4$. Then since $\dim \mathbb{C}[x, y]/(f_1, f_2) = 0$, we conclude f_1 and f_2 have finitely many common zeros in $\mathbb{C}^2$ and thus we can use theorem B.1.1 and theorem B.1.2.

It is clear that $(0, 0)$, $(1, 0)$ and $(0, 1)$ are steady states, so by theorem B.1.1 there can only be one more steady state. Note that each one of the steady states has at least one zero coordinate, so there can be at most one steady state where coexistence occurs.

Using theorem B.1.2, we get the same upper bound by computing the multiplicities:

$$\frac{1}{2}(e(k[NP(f_1f_2)]) - e(k[NP(f_1)]) - e(k[NP(f_2)])) = \frac{1}{2}(4 - 1 - 1).$$

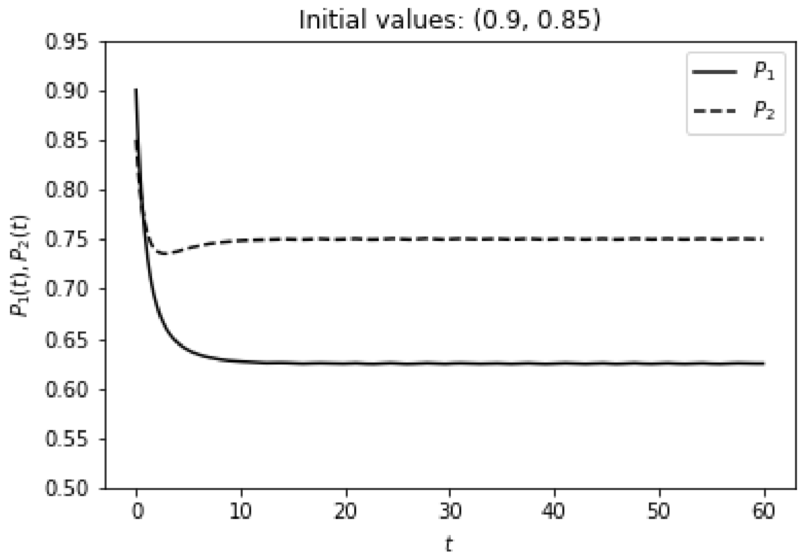


Figure B.1: A numerical solution of the system of differential equations.

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